SOME OPEN PROBLEMS — Math118, O. Knill

ABSTRACT. We summarize some open problems in dynamics. Most have been mentioned during this course. One possibility for a project is to write down a short essay about such a problem.

STABILITY OF EXTERIOR BILLIARDS. Does there exist a billiard table, for which the exterior billiard features an unbounded orbit?

Discussion. A semicircle had been mentioned as a candidate for unstability. One doesn't know the answer also for general polygonal billiards. Smooth strictly convex tables produce bounded orbits (KAM). Some "rational polygons" produce bounded orbits too.

INTEGRABLE EXTERIOR BILLIARDS. Does there exist an integrable smooth convex exterior billiard map different from the ellipse? Integrability means that there should exist a smooth function F for which each set $\{F = c\}$ outside the table is a curve.

Discussion. The problem could be changed by allowing F to be continuous only or requiring F to be realanalytic. For some polygons like the sqare, the billiard is integrable.

MEASURE OF PERIODIC POINTS OF BILLIARDS. Can a smooth billiard table have a set of periodic points which have positive area?

Discussion. It is known that periodic points of period 2 or 3 have zero area. See Rychliks paper of 1998.

MEASURE OF PERIODIC POINTS OF EXTERIOR BILLIARDS. Can a smooth strictly convex exterior billiard table have a set of periodic points which have positive area?

Discussion. I have not seen that question asked and the answer could be easier to find than in the billiard case. Some polygons like the square have a full set of periodic orbits.

MEASURE OF NONCOLLISION SINGULARITIES. what is the measure of noncollision singularities of the Newtonian n-body problem.

Discussion. This is an old problem appearing for example in Simons list of unsolved problems for the 21'st century: The difficulty to figure out all possible configurations leading to noncollision singularities suggest that a completely different approach is necessary.

INTEGRABIILITY AND DYNAMICAL LOGARITHM PROBLEM. Does nonintegrability (in the sense that no smooth invariant function F exists) imply that the dynamical logarithm problem can not be solved efficiently? Specifically, for the nonintegrable map

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2\alpha \\ x + y \end{bmatrix} .$$

on the torus. Assume $\alpha = \pi$, find n such that $T^n(0.5, 0.5)$ is within distance 10^{-1000} of $(0, 0)$.

LATTICE POINTS NEAR PARABOLA. For every $0 \leq \delta < 1$, there exists a positive constant C such that the number $M(n, \delta)$ of $1/n$-lattice points in a $1/n^{1+\delta}$ neighborhood of a parabola satisfies $M(n, \delta)/n^{1-\delta} \to C$ for $n \to \infty$.

Discussion: we know that the result is true for $\delta < 1/3$. We suspect it is true for $\delta < 1$.

WAVE FRONTS AND CAUSTICS ON THE TORUS. Is it true that for a general metric on the two dimensional torus (a bumpy doughnut), the wave front of a point becomes dense on the torus? Is it true that for a nonflat metric, the caustic of a point is dense on the torus?

Discussion. This seems completely unexplored and might be not too difficult. It would be enough to verify that the wavefront K_t contains longer and longer pieces with shrinking curvature.

HILBERTS 16th PROBLEM. Find upper bounds on the number of limit cycles for a differential equation on the plane. A concrete problem: verify that a differential equation

$$\begin{aligned} \frac{d}{dt}x &= p(x, y) \\ \frac{d}{dt}y &= q(x, y) \end{aligned}$$

with polynomials p and q of degree n has only finitely many limit cycles. Find a bound for their number $H(n)$.

Discussion: this is an old and probably very difficult problem.

CELLULAR AUTOMATA. Find closed formulas for the topological entropy of each of the 256 elementary CA in one dimension.

Discussion: there are some automata, for which we know the answer, like for rule 90, the shift. For subshifts of finite type, the entropy is $\log(\kappa)$, where κ is the largest eigenvalue of an integer matrix.

SPEED SPECTRUM OF CA. Given a CA in the plane like life, one can define the speed spectrum as the set of possible speeds, gliders can have. If $T^n(x) = \sigma^m(x)$, then the velocity is $v = m/n$ and the speed $|m|/v$. The speed spectrum of a cellular automaton is a discrete subset of $[0, d]$, where d is the diameter of the CA rule. Find a higher dimensional automaton, where the speed spectrum can be proven to be dense in $[0, d]$.

Discussion: I don't think, this is known in the specific example of "Life". There are many gliders known. "Life" should be rich enough to have a desne speed spectrum.

ENTROPY OF STANDARD MAP. Does the Standard map have a positive average Lyapunov exponent for large λ. More specifically, is it true that for all n,

$$\frac{1}{n} \int_0^{2\pi} \int_0^{2\pi} \log ||A(T^{n-1}(x, y) \cdots A(x, y)|| \, dxdy \geq \log(c/2)$$

where $T(x, y) = (2x + c\sin(x) - y, x)$ and $A(x, y) = dT(x, y)$.

Discussion. The problem had been posed in the sixties by Sinai. One knows that the Lyapunov exponent is positive on horse-shoes which form Cantor sets, but these sets have zero measure.

BOUNDED ORBITS IN SITNIKOV PROBLEM. What is the measure of the set of initial conditions for which the Sitnikov planet stays bounded?

Discussion: there is a continuum of bounded solutions constructed by the horse-shoe construction. The problem is already difficult for maps in the plane, which are explicitely given like the area preserving Henon map.

STABILITY OF SOLAR SYSTEM An ancient problem. Is the solar system stable? More precisely: is the n-body problem with the initial condition given by the position leading to a solution in which orbits stay in a bounded region for all times? How big is the set of initial conditions, for which one has stability?

Discussion. One knows that the measure of initial conditions leading to stability is positive, but one knows no estimates for the size of the stability region.

NONCOLLISION SINGULARITIES WITH 4 BODIES. Does Gervers construction lead to noncollision singularity with four bodies?

Discussion. Gervers construction works in the plane. Xias construction needed a lot of mathematical resources. It is likely that also this problem is very difficult.

CONSTRUCTING LATTICE POINTS CLOSE TO PARABOLA. Can one solve the dynamical logarithm problem efficiently for for maps obtained from polynomials.

Discussion: if it were, one could factor integers fast.

LYAPUNOV EXPONENT FOR DOUBLE PENDULUM. Is the Lyapunov exponent positive for the double pendulum (equal mass and equal length of the legs). The same question can be asked for other Hamiltonian systems for which the flow leaves a bounded region invariant like the Henon-Heils system.

Discussion. There are many Hamiltonian systems of two degree of freedom, where chaos is observed. While it is often possible to construct a Cantor set on which the dynamics is chaotic, the question is whether one can establish this kind of motion on a set of positive measure.

LOCAL CONNECTIVITY OF MANDELBROT SET. is the Mandelbrot set locally connected?

Discussion. This is the holy grail in complex dynamics.

CHAOTIC SMOOTH CONVEX BILLIARD. Find a smooth convex billiard table for which the Lyapunov exponent is positive on a set of positive area.

Discussion. One could try to smoothen out the stadium but any of these attempts produces stable periodic orbits which distroy ergodicity. Also invariant curves near the boundary will prevent ergodicity. One has to prove coexistence of stable and unstable behavior.

CHAOTIC THREE BODY PROBLEM. Is the Lyapunov exponent positive on a set of positive area for some restricted three body problem.

Discussion. By constructing horse shoes, one can get Cantor sets of zero area, for which the Lypapunov exponent is positive.

NORMALITY OF PI. Are the digits of Pi uniformly distributed? In other words, is π normal. The same question can be answered for other numbers like $\sqrt{2}$.

Discussion. Some doubts have been raised that π produces good random numbers: http://news.uns.purdue.edu/UNS/html4ever/2005/050426.Fischbach.pi.html

3N+1 PROBLEM. Is $1 \to 4 \to 2$ the only attractor of the Collatz map $T(x) = \left\{ \begin{array}{c} x/2, \text{x even} \\ 3x+1, \text{x odd} \end{array} \right.$.

Discussion. The mathematical tools seem not to catch. The best hope for success is probably to find a periodic orbit different from the trivial one. Heuristic arguments show however that the "chance" for success is small. As higher periodic orbits you look for, they are more and more "unlikely".

A LATTICE POINT PROBLEM. The map $T(x) = (3/2)x \; mod \; 1$ is related to a lattice point problem for a function on the real line. It is conjectured that the orbits of T are uniformly distributed modulo 1.

Discussion. For $T(x) = 2x$ mod 1, the distribution is uniform for most initial points x.

HOMOCLINIC POINTS AND INTEGRABILITY. Assume that a smooth map T on the plane has a hyperbolic fixed point with a transverse homoclinic point. Prove that there is no smooth function F which is invariant for which $\{F = c\}$ is either a curve or a finite set.

Discussion. This might be relatively easy to settle. The existence of a transverse homoclinic point produces a horse shoe as we have seen. The function is constant on that horse shoe as well as on the union of the invariant stable and unstable manifolds. While the horse shoe has dimension bigger than 0, it could be part of a complicated level set which is a union of arcs. But unlike realanalytic functions, smooth function can have complicated level sets.

BLANCHARD PROBLEM. Does every transitive automaton have a dense set of periodic points?

Discussion: An automaton T is called transitive, if it has a dense orbit in X. We have seen that the shift is transitive. We also have seen that the shift has a dense set of periodic points. Francois Blanchard writes: "The answer, positive or negative, is a necessary step before on understands the meaning of chaos in the field. " Source: This problem can be found in Michael Misiurewicz list of open problems in dynamical systems (http://www.math.iupui.edu/ mmisiure/open)

COHEN MAP. Does the hyperbolic point of period 14 found by Hubbard in the map

$$T(x, y) = (\sqrt{x^2 + 1} - y, x)$$

have stable and unstable manifolds which intersect transversly?

Discussion: I had been asked as an undergraduate to make experiments with this map to see whether it is integrable or not - and did not find any signs of non-integrability. Marek Rychlik told me in 1998, that numerical experiments by John Hubbard revealed a hyperbolic periodic orbit of period 14: $(x, y) = (u, u)$ with $u = 1.54871181145059$. The largest eigenvalue of $dT^{14}(x, y)$ is $\lambda = 1.012275907$. The existence of a hyperbolic point of such a period makes integrability unlikely since homoclinic points might exist, but it is not impossible. It is difficult to find other hyperbolic periodic points. An other indication for non-integrability is a result of Rychlik and Torgenson who have shown that this map has no integral given by algebraic functions.