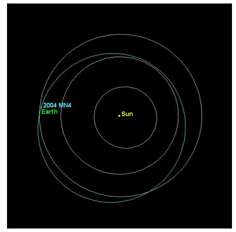
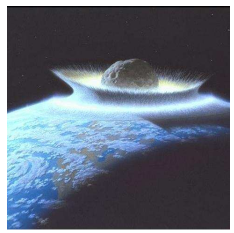
RESTRICTED 3 BODY PROBLEMS	Math118, O. Knill

ABSTRACT. The Newtonian 3 body problem can exhibit chaos. The simplest situation is when the third body moves in the time dependent potential of a binary system but itself does not influence the motion of the binary system. A first example is the **Sitnikov problem**, where one can establish the existence of a **horse shoe** which leads to a in general **chaotic calendar** for inhabitants of the Sitnikov planet. An other example is the circular planar restricted three body problem which leads to cases, where one has an area preserving map on a region with finite area. It is also a historically important example because some results in ergodic theory like Poincare recurrence and topology like fixed point theorems were developed with the three body problem in mind.

RESTRICTED THREE BODY PROBLEMS. The **restricted 3-body problem** deals with the situation, where one of the three bodies has a neglectable mass, and moves under the influence of the two other bodies which evolve according to Keplers law. Lets call here the two heavy bodies the **double star binary system** and the third body the **planet**.



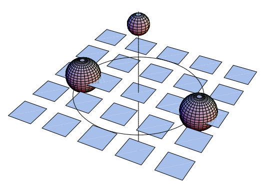
ASTEROID 2004 MN4 IMPACT RISK? In December 2004, Asteroid 2004 MN4 was given a 1/233 chance, then a 1/38 chance to hit the earth in April 13, 2029. Despite numerological support for bad luck like 2+0+2+9=13 and 1+3=4=shi also means "death" in Japanese, subsequent observations have shown that there will be no impact in 2029. It will pass by the Earth at a distance of between 15'000 and 25'000 miles, about a tenth of the distance between the Earth and the Moon and be so close that it can e seen with the naked eye. The change of orbit might put 2004 of a collision course in 2034, 2035 or 2036. One will know more in 2029.



SITNIKOV PROBLEM. The **Sitnikov problem** deals with the situation, where the double star system moves in the xy-plane and the planet is on the z-axes. Both stars have equal mass m normalized to $m = 1/2$ and move on elliptic orbits, where the center of mass is at rest. The third body has no mass. Its z coordinate satisfies the **Sitnikov differential equation**

$$\frac{d^2z}{dt^2} = -\frac{z}{(z^2 + r(t)^2)^{3/2}} ,$$

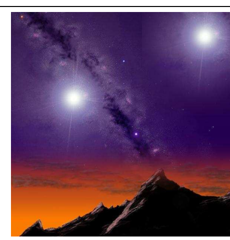
where $r(t)$ is the distance of a sun to the origin at time t. By normalizing time, we can assume that $r(t)$ has period 2π. For small values of the eccentricity ϵ of the ellipse, one has $r(t) = \frac{1}{2}(1 - \epsilon\cos(t)) + O(\epsilon^2)$.



SITNIKOV YEAR. A **Sitnikov year** is the time it takes to return to the xy-plane, the summer position on Sitnikov planet. Winter is when the planet has the maximal distance to the stars. The inhabitants on "Sitnikov" know to measure time and count the number of **Sitnikov days** in one Sitnikov year k by

$$s_k = [(t_{k+1} - tk)/2\pi] .$$

Far away from the double star system, a winter day could look as in the picture to the right.



A CHAOTIC CALENDAR.

THEOREM (Sitnikov-Moser) For sufficiently small eccentricity $\epsilon > 0$, there exists an integer m such that for any sequence $s_1, s_2, ...$ of of integers $s_k \geq m$, there exists a solution of the Sitnikov differential equation for which year k has s_k days.

REMARKS. One can also allow $s_k = \infty$ in which case, the planet would escape for ever, or the solar binary system could capture an orbit which stays bounded for ever. The proof of the theorem relies on the horse shoe construction and is robust. The result therefore holds also for planets with small positive mass. The result can be shown to be true for all $0 < \epsilon < 1$ except a discrete set of values.

Most orbits in this dynamical system go to infinity. It is not quite clear what the **filled in Julia set** is, the points which stay bounded for all times. Sitnikov-Moser theorem constructs a Cantor set of points which stay bounded for ever. It is not excluded that there are some stable elliptic periodic points. Numerical experiments suggest that such stable periodic points exist but I have not seen a proof. The stability problem is in nature similar to the one for the quadratic Henon map in the plane and depends on subtle Diophantine properties which have to be satisfied for the periodic points. We expect for most parameter values ϵ a set of positive area stays bounded. This could be good news for Sitnikov inhabitants.

The bad news is that these regions might be very small and a small disturbance - for example by an asteroid - could free the Sitnikov planet and send its inhabitants to a deadly eternal winter ride. One of the last pictures taken from that escaping planet could look as the picture above.

TO THE PROOF (Moser 1973).

Look at the **Poincare return map** to the plane with polar coordinates $(r, \phi) = (|v|, t)$, where v is the velocity of the planet and t *mod* 2π is the time given by the suns clock. $t = 0$ corresponds to the moments, when the suns are closest to the z axes. The return map is defined in a simple closed region D_0. Outside this region, the orbit escapes. Here is an outline of the proof. The details are quite technical and can be found in Mosers book.

(0) The return map T_ϵ maps D_0 into $D_1 = \rho(D_0)$, where ρ is the reflection $(v, t) \to (v, -t)$. The map T_ϵ is area preserving: the area element $2vdvdt = dE\ dt$ is preserved.

(i) For small enough ϵ, the boundaries of D_0 and D_1 are smooth curves which intersect transversely. The proof of this fact is done by writing the right hand side of the Sitnikov equations as a power series in ϵ and neglecting ϵ^2 and larger terms. This computation from perturbation theory allows to establish that the angle between the boundary curves becomes nonzero.

(ii) For $\epsilon = 0$, the map T_0 is integrable and of the form

$$T_0 \begin{bmatrix} v \\ t \end{bmatrix} = \begin{bmatrix} v \\ t + f(v) \end{bmatrix}$$

where $f(v) \to \infty$ if $v \to 2$. The differential equation is in this case

$$\ddot{z} = \frac{-z}{(z^2 + 1/4)^{3/2}} .$$

This is an integrable system: indeed, the energy

$$E = \frac{1}{2}\dot{z}^2 - \frac{1}{\sqrt{z^2 + 1/4}} \geq -2$$

is conserved and the map leaves its level curves of E invariant. The origin is a fixed point, each circle gets rotated and the rotation becomes faster and faster until the boundary $E = 0$ is reached. In physical terms, this means that if we start with a larger initial velocity, it takes longer to return.

(ii) There are horse shoes arbitrarily close to the boundary of D_0. This is a consequence of i and ii and will be explained in class. (needs a good picture)

PLANAR CIRCULAR THREE BODY PROBLEM. The **planar restricted 3-body problem** deals the situation, where one of the three bodies has neglectable mass, but moves under the influence of two other bodies which evolve along circles according to Keplers law. An example is the motion of the moon in the influence of the earth and sun. A second example is the motion of an asteroid under the influence of the sun and Jupiter, the second largest body in our solar system. An other example is the motion of a planet in a binary star system.

ROTATING COORDINATE SYSTEM. Assume $\vec{y} = R(\omega t)\vec{x}$, where $R(\alpha)$ is a rotation in the plane with angle α. We can write $R(\omega t) = e^{A\omega t}$, where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

LEMMA. In the rotating coordinate system

$$\frac{d^2}{dt^2}\vec{y} = R\frac{d^2}{dt^2}\vec{x} + 2A\omega R\frac{d}{dt}\vec{x} - R\omega^2\vec{x}$$

one observes additionally to the **rotated forces** also a **centrifugal force** and a velocity dependent **Coriolis forces**.

PROOF. Differentiating twice the identity $y = Rx$ using $\dot{R} = \omega AR$ gives $\dot{y} = \dot{R}x + R\dot{x} = \omega AR\vec{x}$ and $\ddot{y} = \omega^2 A^2 R\vec{x} + 2AR\dot{x} + R\ddot{x}$. Because $A^2 = -1$, this gives the equation in the lemma. The same calculation in coordinates: $\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = R\begin{bmatrix} \dot{x}_1 - \omega x_2 \\ \dot{x}_2 + \omega x_1 \end{bmatrix}$ and $\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = R\begin{bmatrix} \ddot{x}_1 - \omega^2 x_1 - 2\omega x_2 \\ \ddot{x}_2 - \omega^2 x_2 + 2\omega x_1 \end{bmatrix}$, where $R = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}$. Remark. The same computation can be done in three dimensions, where both the centrifugal and Coriolis forces can be expressed using cross products.

THE EQUATIONS OF THE PLANAR CIRCULAR 3-BODY PROBLEM. Two stars of mass $m_1 = \mu, m_2 = 1-\mu$ move on circular orbits along their center of mass. Going into a rotating **inertial coordinate system** (Keplers 3. law implies from zero eccentricity uniform rotation), in which the stars are fixed at the points $(1-\mu,0), (-\mu,0)$, the equations of motion become

$$\frac{d}{dt}x_k = E_{y_k}, \frac{d}{dt}y_k = -E_{x_k} ,$$

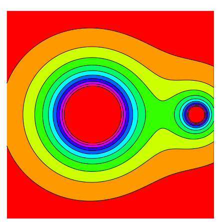
where $E = \frac{1}{2}(\dot{y}_1^2 + \dot{y}_2^2) + 2x_2y_1 - 2x_1y_2 - \frac{\mu}{r_1} - \frac{(1-\mu)}{r_2}$ is the Hamilton function. Here $r = \sqrt{x_1^2 + x_2^2}$ is the distance of the planet to the origin, $r_1 = \sqrt{(x_1+\mu-1)^2 + x_2^2}$ and $r_2 = \sqrt{(x_1-\mu)^2 + x_2^2}$ are the distances from the planet to the two stars. We can decompose $E = (\dot{x}_1^2 + \dot{x}_2^2)/2 - U(x_1,x_2)$ with $U = \frac{1}{2}r^2 + \frac{\mu}{r_1} + \frac{(1-\mu)}{r_2}$. The function E is called the **Jacobi integral**. It contains $\frac{1}{2}r^2$ called **centrifugal potential** and $\dot{x}_1^2 + \dot{x}_2^2$, the **Coriolis potential**. How did we get that? The Newton equations in the rotating coordinate system are according to the previous lemma:

$$\ddot{x}_1 - 2\dot{x}_2 = \frac{\partial}{\partial x_1}U$$
$$\ddot{x}_2 + 2\dot{x}_1 = \frac{\partial}{\partial x_2}U$$

After multiplying the first equation with $\dot{x}_1$ and the second with $\dot{x}_2$, addition gives $\dot{x}_1\ddot{x}_1 + \dot{x}_2\ddot{x}_2 = \frac{\partial}{\partial x_1}U\dot{x}_1 + \frac{\partial}{\partial x_2}U = \dot{U}$ so that $E = (\dot{x}_1^2 + \dot{x}_2^2)/2 - U$ is conserved. Introducing $y_1 = \dot{x}_1 - x_2, y_2 = x_1 + \dot{x}_2$ leads to the Hamilton equations at the top of this box.

What is the deal? We started with the Newton equations $\ddot{y}_i = \frac{\partial}{\partial x_1}W$ and ended up with a system looking more complicated. But it is not! In the original coordinates, the potential W is time dependent! Especially, there was no energy conservation. Going into the rotating coordinate system led us to a Hamiltonian system with a preserved quantity, the Jacobi integral.

HILLS REGION. Assume $E = c_1$ and $c < c_1$. The regions $U(x_1,x_2) = c$ bound regions in the (x_1,x_2) plane called **Hills regions**.



LEMMA. If (x_1,x_2) is in a Hills region $U \geq c$, then $(x_1(t),x_2(t)$ is in the Hills region for all times.

For large c, these regions consist of three parts. Two in the neighborhood of the two stars (satellite bound by one of the bodies) and one far away (asteriod encircling both). They define an allowed region in which the planet can stay. A large c corresponds to the case, where one is either close to one of the stars with large gravitational potential or very far away, with large centrifugal potential.

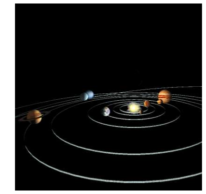
RECURRENCE. The energy surfaces $E = c$ are invariant as are the sets $\{(x_1,x_2) \mid a \leq -E(x_1,x_2) \leq b\}$ for $a < b$. If $c < c_1$, then

$$G = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - E > c_1 > c .$$

So, (x_1,x_2) stays in a bounded region. Also (x_1,x_2,y_1,y_2) stays in a bounded set. The differential equation preserves the four dimensional volume. When normalizing the volume to 1, we obtain a probability space. The time 1 map is a measure preserving map on that space and Poincares recurrence theorem applies.
There is a subtlety with this argument which has to be mentioned: Not all solutions in the finite region have a global solution. There are initial conditions, in which the planet crashes into one of the suns but these cases can be shown to have zero volume.

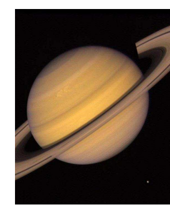
CHAOS IN THE SOLAR SYSTEM. Chaos in the solar system has been measured at different places:

1) The solar system itself is weakly chaotic. The Lyapunov exponent has been measured to be very small $2.8 \cdot 10^{-15}$. For Pluto the Lyapunov exponent had been measured $7 \cdot 10^{-16}$. Numerical experiments have also been done with other parameters. The heliocentric distance for outer planets would behave much more erratically, if the sun would have 1/3 less of its current mass, suggesting that some of the outer planets like Neptune or Uranus would escape in such a case. For our solar system, it looks as if one can not predict the trajectory of the earth for time periods exceeding 100 Million years. More precisely, the uncertainty of 1 km in the initial condition could lead to an uncertainty of the order of 1 astronomical unit in 100 Million years. Numerical simulations of the solar system have been done for time intervals reaching 35 billion years.



2) Many **comets and asteroids** in the solar system have irregular orbits. Numerical experiments have been done for example in the case of the asteroid **Chiron**. To measure sensitive dependence on initial conditions, one starts integrating with various close initial conditions and looks at the outcome. Chiron will undergo several close approaches to planets. One estimates a 1/8 chance that Chiron will eventually leave the solar system. Other objects have an other fate. The comet **Shoemaker-Levy 9** had a spectacular impact with Jupiter in July 1994 after having been disrupted by a close Jupiter approach in 1992.

3) The tumbling of Saturns little moon **Hyperion**. Most satellites in the solar system are in synchronous rotation, keeping one face towards the planet. Hyperion has an irregular shape and is known to tumble erratically in its orbit. The **Cassini spacecraft** will fly past this moon later this year, on September 26, 2005. The Lyapunov exponent of the irregular tumbling motion has been been measured to be of the order 10^{-7}.



4) The motion of charged particles in a magnetic dipole field has been shown to be chaotic. Brown has constructed a horse shoe for the return map. The dynamics can be reduced to a relatively simple Hamiltonian system

$$H(q_1,q_2,p_1,p_2) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(\frac{1}{q_1} - \frac{q_1}{(q_1^2+q_2^2)^{3/2}})^2$$

called the **Stoermer problem**. The dynamics of charged particles in the **van Allen belts** can explain the **aurora Borealis**.

For the Lyapunov exponent data on this box, we the sources:
P. Gaspard: "Chaos Scattering and Statistical mechanics", 1998
I. Peterson: "Newtons Clock: Chaos in the solar system", 1993
C.D. Murray and S.F. Dermott: Solar system dynamics", 2001
D. Goroff: Editorial introduction article in "New Methods of Celestial Mechanics by H. Poincare".
K. Zyczkowski "On the stability of the Solar system".
For the planar 3 body problem, we followed Siegel-Moser. Sitnikovs problem is treated in detail in Mosers 1973 book.