NOTIONS IN COMPLEX DYNAMICS

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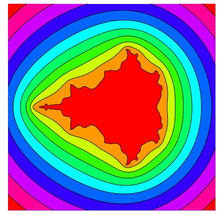
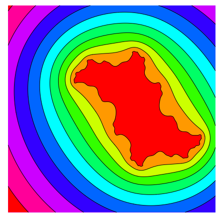
ABSTRACT. This page summarizes some definitions in complex dynamics and gives a brief jumpstart to some notions in complex analysis and topology.

MANDELBROT SET. $f_c(z) = z^2 + c$ is called the quadratic map. It is parametrized by a constant c. The set M of parameter values c for which $f_c^n(c)$ stays bounded. In the homework you seea that $M = \{c, |f_c^n| \leq 2$ for all $n\}$. With $G(c) = \lim_{n\to\infty} \log |(f_c^n(c))^{1/2^n}|$ one can also say $M = \{c \mid G(c) = 0\}$. The level curves of G are **equipotential curves**: if you would charge the Mandelbrot set with a positive charge, $G(z) = c$ is the set of points where the attractive force of an electron to the set is the same. By definition, M is closed. Douady-Hubbard theorem tells it is connected. That M is **simply connected** is much easier to see: it follows from the **maximum principle** that the complement of M is connected.



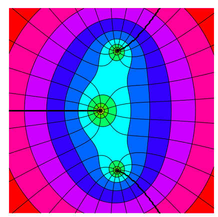
JULIA SET. The set of complex numbers z for which $f_c^n(z)$ stays bounded is called the **filled in Julia set** K_c. It is the set of z for which the function $G_c(z) = \lim_{n\to\infty} \log |(f_c^n(z))^{1/2^n}|$ is zero. Its boundary is called the **Julia set**. The Julia set can be a smooth curve like in the case $c = 0$ or for $c = -2$ but it is in general a complicated fractal. It is known that the Julia set J_c is the closure of the repelling periodic points of f_c. It is also known that f_c restricted to J_c is chaotic in the sense of Devaney. The complement of J_c is called the Fatou set F_c. The bounded components of F_c are called **Fatou components**.



COMPLEX MAPS. A complex map f can be written as a map in the real plane $f(x+iy) = u(x,y) + iv(x,y)$. The derivative at a point z_0 is defined as the complex number

$$a = f'(z) = \lim_{w\to 0} (f(z+w) - f(z))/w \, .$$

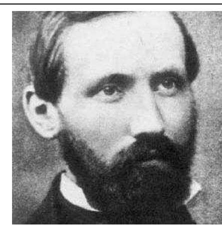
If the derivative exists at each point in a region U and f' is a continuous function in U, the map f is called **analytic** in U.



CAUCHY-RIEMANN. Since the linearization of f at z_0 is the map $z \to az$ which is a rotation dilation and the linearization of f is the Jacobean

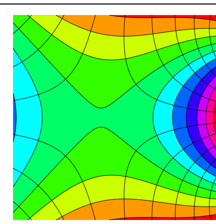
$$A = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} ,$$

we must have $\boxed{u_x = v_y, u_y = -v_x}$ (A rotation matrix has identical diagonals and antidiagonals of opposite signs and this property is preserved after multiplying the matrix with a constant) . These two equations for u, v are called **Cauchy-Riemann differential equations**.



CONFORMALITY. If $a \neq 0$, then angles are preserved because both rotations and dilations preserve angles. Therefore the rotation dilation $z \to az$ preserves angles. If $f'(z)$ is never zero in a region U, the map f is called **conformal** in U. In that case, it maps U bijectively to $f(V)$ and preserves angles. Angle preservation is useful in cartography or computer graphics.

HARMONICITY. From the Cauchy-Riemann equations follows $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$. Therefore, the real and imaginary part of f are **harmonic functions**. The mean value property $\int_{|w-z|=r} u(w(t)) \; dt = u(z)$ and $\int_{|w-z|=r} v(w(t)) \; dt = v(z)$ for harmonic functions can be written as $\int_{|w-z|=r} f(w(t)) \; dt = f(z)$.



TAYLOR FORMULA. Because $df(w(t))/dt = f(x + r\cos(t) + i(y + r\sin(t)) = f'(w)(r\cos(t) + ir\sin(t)) = f'(w)(z-w)$, this can be rewritten as $\int_{|w-z|=r} f'(w(t))dt/(z-w) = f(z)$. This is the **Cauchy integral formula**. Since we can differentiate the left hand side arbitrarily often with respect to z, this proves that an analytic function is arbitrarily often differentiable and $f(w)/(z-w)$ has the n'th derivative $\frac{f(w)}{n!(z-w)^{n+1}}$, we get

$$f(w) = \sum_n \frac{f^{(n)}(z)(w-z)^n}{n!}$$

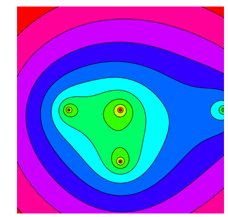
which is the familiar **Taylor formula** if f is real.

CAUCHY THEOREM. The Cauchy Riemann equations also prove the **Cauchy formula**. If C is a closed curve in simply connected region U in which f is analytic, then

$$\boxed{\int_C f(z)dz = \int f(z(t))z'(t) \; dt = 0}$$

because the later is the line integral of $F(x,y) = (-v(x,y), u(x,y))$ and **Greens theorem** in multi-variable calculus shows that $\text{curl}(F) = \text{curl}((-v,u)) = (u_x - v_y) = 0$. In other words, the vector-field $F(x,y) = (-v(x+iy), u(x+iy))$ is conservative.

FIXED POINTS. Because the eigenvalues of the rotation dilation A come in complex conjugate pairs, the fixed points or periodic points can not be hyperbolic. Fixed points are either stable sinks, or unstable sources elliptic, conjugated to a rotation. For example, the fixed points of $f(z) = z^2 + c$ are $(1 \pm \sqrt{1-4c})/2$ and the linearization at those points is $df(z) = (1 \pm \sqrt{1-4c})z$



TOPOLOGY. Here are some topological notions occuring in complex dynamics:
OPEN. A set U in the plane is called **open** if for every point z, there exists $r > 0$ such that $B_r(z) = \{w \mid |w-z| < r\}$ is contained in U. One assumes the empty set to be open. The entire plane is open too.
CLOSED. A set U in the plane is **closed**, if the complement of U is open. The entries plane is closed.
INTERIOR. The **interior** of a set U is the subset of all points z in U for which there exists $r > 0$ such that $B_r(z) \subset U$. If a set is open, then it is equal to its interior.
CLOSURE. The **closure** of a set U is the set of all points which are limit points of sequences in U. It is the complement of the interior of the complement of U. If a set is closed, then U is equal to its closure.
BOUNDARY. The boundary of a set U is the closure of U minus the interior of U. The boundary of a closed set without interior is the set itself.
SIMPLY CONNECTED. A set A is **simply connected**, if every closed curve contained in A can be deformed to a point within A. A simply connected subset of the plane has no "holes".
CONNECTED. A set A is called **connected** if one can not find two discjoint open sets U, V such that $A \cap U \neq \emptyset$, $A \cap V \neq \emptyset$.

$$\boxed{\text{A set } A \text{ is connected if and only if the complement is simply connected.}}$$

To verify that the complement of M is simply connected, one finds a smooth bijection of the complement of the unit disc with the complement of M. The bijection is given by $\Phi(c) = \lim_{n\to\infty}(f_c^n(c))^{1/2^n}$. The Mandelbrot set M is connected as well as simply connected. The Julia sets J_c are connected, if c is in M.
COMPACT. A subset of the complex plane is called **compact** if it is closed and bounded. A sequence in a compact set always has accumulation points. The Mandelbrot set as well as the Julia sets are examples of compact sets.
PERFECT SETS. A subset J in the complex plane is **perfect** if it is closed and every point z in J is accumulation point of points in $S \setminus z$. Perfect sets contain no isolated points.
NOWHERE DENSE. A subset J in the complex plane is **nowhere dense** if the interior of its closure is empty. A Julia set J_c is nowhere dense if c is outside the Mandelbrot set.
CANTOR SET. A perfect nowhere dense set is also called a **Cantor set**. An example is the **Cantor middle set**. A Julia set J_c is a Cantor set if c is outside the Mandelbrot set.