MATH 118
PROBLEM SET 8

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Exercise 1. (a) Show that the matrix $A = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$ generates an asymptotically stable flow e^{tA}.

(b) Sketch the phase portrait of $\frac{dx}{dt} = Ax$.

(c) Show that it is false that $\left|e^{tA}x\right| \leq |x|$ for all $t > 0$ and $x \in \mathbb{R}^2$, where $|x|$ is the standard Euclidean norm.

Exercise 2. Suppose e^{tA} and e^{tB} are both contractions on $\mathbb{R}^n$. Show $E^{t(A+B)}$ is a contraction too, if $AB = BA$. What if $AB \neq BA$?

Exercise 3. Verify the theorem from class for the matrix

$$A = \begin{bmatrix} 3 & 2 & 5 \\ -1 & -2 & -5 \\ 1 & -4 & -1 \end{bmatrix}$$

by showing that the the flow solving $\frac{dx}{dt} = Ax$ is topologically conjugate to the system $\frac{dy_i}{dt} = \lambda_i y_i$ with $\lambda_i = \pm 1$ for $i = 1, 2, 3$.

Exercise 4. Sketch the four topologically distinct types of phase portraits that can occur at a hyperbolic fixed point of a flow on $\mathbb{R}^3$.

Exercise 5. Consider the nonhomogeneous linear system $\frac{dx}{dt} - Ax = b(t)$, where $x \in \mathbb{R}^2$ and $b\colon \mathbb{R} \to \mathbb{R}^n$. Find a particular solution of the form $x(t) = e^{tA}z(t)$ for an appropriate $z\colon \mathbb{R} \to \mathbb{R}^n$. This is called the method of "variation of constants." Show that the general solution with initial condition $x(0) = c$ has the form

$$x(t) = \int_0^t e^{(t-s)A} b(s)\, ds + e^{tA}c.$$

Use this to solve

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$$

for the solution other solutions tend to as $t \to \infty$.

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