MATH 118: SPRING 1999
COMPUTER PROJECT 1: NEWTONIAN ITERATION

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In this project, we'll use *Mathematica* to apply Newton's method to some functions. We will see that it is very helpful in finding roots of functions, but not always so helpful in finding the nearest one!

The idea will be that given a function f, we can create the map $N_f(x) = x - \frac{f(x)}{f'(x)}$, and investigate which root (if any) the sequence $\left(N_f^n(x)\right)_{n\geq 0}$ converges to. We can see this graphically to propose a conjecture, then attempt to prove it mathematically.

Let's begin with a simple quadratic. Let $f(x) = x^2 - 1$. We already know the roots of f; they are ± 1. How does Newton's method respect this? For instance, if we start near one of the roots, do we converge to it under iterates of N_f?

Exercise 1. Code (define) the function N_f which performs the Newton's map for f. Plot f and N_f. Where are the critical points and poles of N_f?

Now we make use of *Mathematica*'s iteration and data-manipulation properties. To determine the "limiting" behavior of N_f, we have to approximate by some large iterate.

Exercise 2. Explain what you think the expression

```
Table[{x,Nest[Nf,x,10]},{x,-2,2,0.1}]
```

should do. Evaluate the expression then use `ListPlot` to plot the list of points.

What you see should be a set of horizontal lines at elevations equal to whichever root those points are converging to under iteration, either plus or minus one. By changing some of the numbers in the previous expression, we can "zoom in" on any part of the real line. Of particular interest is the point where we see a difference on either side. Notice that a small but positive x appears to converge to 1, but a small negative x converges to -1.

Exercise 3. Prove that if $x < 0$, then $N_f^n(x) \to -1$, and if $x > 0$, then $N_f^n \to 1$.

So far, so good. Let's boot the complication up by one degree. Let $g(x) = x^3 - x$.

Exercise 4. Repeat Exercise 1 for g.

Exercise 5. Repeat Exercise 2 for g.

You should see something a lot different than what you saw in Exercise 2. In a neighborhood of each of the roots of g, we get a *basin of attraction*, i.e, a place where everything iterates towards the root. However, on the boundaries between basins, we see lots of complicated behavior. Change the parameters of your code to "zoom in" on these boundaries.

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Exercise 6. Produce plots of N_g^{10} on smaller and smaller scales near any of these boundaries.

To make things really hard, let's try the function $\sin x$.

Exercise 7. Repeat Exercise 1 for sin.

Exercise 8. Repeat Exercise 2 for sin.

Focus your attention on a neighborhood of $x = \frac{\pi}{2}$. The closer you get to $\frac{\pi}{2}$, the more varied the values seem to become. Why is this?

Exercise 9. Prove that for any neighborhood I of $\frac{\pi}{2}$, and root y of sin, there is an $x \in I$ such that $N_{\sin}^n \to y$.

We close with a bit of fun, more exciting graphics. Recall that we can do Newton's method in the complex plane just as easily as on the real line. The geometric motivation breaks down, but the naïve algorithm still works.

Try $h(z) = z^3 - 1$. The three roots of h are the three cube roots of unity. Let's iterate Newton's method in a rectangular region of $\mathbb{C}$ and see what happens.

Exercise 10. Code the function N_h. Explain what you think the expression

```
data = Table[Nest[Nh,x + I y,10],
             {y,1.2,-1.2,-0.1},
             {x,-1.2,1.2,0.1}];
```

will do. Try it out and then perform `TableForm[data]` to verify your answer.

Now let's crudely take advantage of *Mathematica*'s color capabilities.

Exercise 11. Evaluate the expressions

```
RootColor[z_] := Hue[Arg[z]/(2Pi)]
Show[Graphics[RasterArray[Map[RootColor,data,{2}]]],
     AspectRatio->1]
```

What does this graphically represent? Play around with the parameters to get graphics of finer and finer detail.

Your completed project should consist of written work and *Mathematica* output. Consider it like a laboratory report, with the computer code your data. Edit the latter to make sure it's clear to the grader what you're trying to do. If you want, you can take advantage of the Format:Style menu to put documentation directly into the *Mathematica* notebook.

This project is due Thursday, March 18.