MATH 118 : SPRING 1999
SOLUTIONS TO PROBLEM SET 3

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Exercise 1. (a) *Suppose* $f\colon \mathbb{R} \to \mathbb{R}$ *is continuous and* $f(x_0) > 0$. *Show that* $f(x) > 0$ *in a neighborhood of* x_0.
(b) *Explain how and why we used this result in one of our proofs.*

Proof. (a) There exists a $\delta > 0$ such that $|f(x) - f(x_0)| < f(x_0)/2$ when $|x - x_0| < \delta$. Thus, for x in this neighborhood,

$$\begin{aligned} -\frac{f(x_0)}{2} &< f(x) - f(x_0) < \frac{f(x_0)}{2} \\ \implies \frac{f(x_0)}{2} &< f(x). \end{aligned}$$

(b) We used this in the proof of the Attracting Fixed Point Theorem. We said that $|f'(p)| < 1$ means that $f'(x)$ is in absolute value less than one on a neighborhood of p.

□

Exercise 2. *Let* X *be a complete metric space.*
(a) *Show that if* $T\colon X \to X$ *is a contraction, then* T^n *is, too.*
(b) *Give an example of a continuous* $T\colon X \to X$ *that is not a contraction but* T^n *is a contraction for some* n.
(c) *For such a map, conclude that there is an* $x \in X$ *fixed by* T^n. *What (prime) period must* x *have under* T *and why?*

Proof. (a) Let T have contraction ratio $\lambda < 1$. Then if x and y are in X, we have

$$\begin{aligned} d\big(T^n(x), T^n(y)\big) &\le \lambda d\big(T^{n-1}(x), T^{n-1}(y)\big) \le \dots \\ &\le \lambda^n d(x,y). \end{aligned}$$

(b) Define $T\colon \mathbb{R} \to \mathbb{R}$,

$$T(x) = \begin{cases} 1 & \text{if } |x| \le 1; \\ 2 - |x| & \text{if } 1 \le |x| \le 2; \\ 0 & \text{otherwise.} \end{cases}$$

Then T^2 is the constant function 1, certainly a contraction.

(c) T^n must have a fixed point p by the Contraction Mapping Theorem applied to T^n. But notice this: let $q = T(p)$. Then

$$T^n(q) = T^n\big(T(p)\big) = T^{n+1}(p) = T\big(T^n(p)\big) = T(p) = q,$$

so q is fixed by T^n. Since T^n was a contraction, $q = p$. This means that p is actually fixed by T!

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□

Exercise 3. (a) *See by direct calculation where the equation* $x^2 + x + 3u + 1 = 0$ *defines* x *implicitly as a function of* u.
(b) *Explain whether your answer agrees with the Implicit Function Theorem's.*

Proof. Solving the quadratic in x, we have

$$x = \frac{-1 \pm \sqrt{-3 - 12u}}{2}.$$

So we have a real solution as long as $u < -1/4$. The Implicit Function Theorem says that at any point on the graph, we can write x as a function of u as long as $\partial P/\partial x \neq 0$. Of course, $\partial P/\partial x = 2x + 1$, which vanishes precisely at the point $(-1/2, -1/4)$. □

Exercise 4. *Suppose* T *maps a complete metric space* (X, d) *to itself and satisfies* $d(Tx, Ty) \leq d(x, y)$ *for all* x *and* y *in a closed ball* $Y = \{\, x \mid d(x, x_0) \leq r \,\}$. *Give a bound on* $d(x_0, Tx_0)$ *that allows you to state a prove a fixed point theorem for* T.

Proof. If we were able to say that T preserves Y and is a contraction when restricted to Y, then we'd be done; we could apply the Contraction Mapping Theorem to $T\colon Y \to Y$ (note that a closed subspace of a complete space is complete in its subspace topology).

So to show that T preserves Y, let $y \in Y$. So $d(y, x_0) \leq r$. Then

$$\begin{aligned} d\big(T(y), x_0\big) &\leq d\big(T(y), T(x_0)\big) + d\big(T(x_0), x_0\big) \\ &\leq \lambda r + d\big(T(x_0), x_0\big), \end{aligned}$$

so a suitable bound for $d\big(T(x_0), x_0\big)$ is $(1 - \lambda)r$. □

Exercise 5. (a) *One non-Newtonian method for solving* $f(x) = x^3 + x - 1 = 0$ *might be to iterate* $g(x) = (1 + x^2)^{-1}$. *Why? Try a few steps beginning with* $x_0 = 1$. *Can you estimate the errors, i.e., the distance from successive iterates from a solution?*
(b) *Show that another method would be to iterate* $g(x) = x^{1/2}(1 + x^2)^{-1/2}$. *Chose* $x_0 = 1$, *calculate a few steps, and explain what is happening.*

Proof. (a) Cleary a fixed point of g is a root of f. Now by calculus, $g'(x) = -\frac{2x}{(1+x^2)^2}$ has a global maximum at $x = -\frac{1}{\sqrt{3}}$ and a global maximum at $x = \frac{1}{\sqrt{3}}$. In fact, the maximum and minimum values are $\pm\frac{3\sqrt{3}}{8}$, which are less than one in absolute value. So $|g'(x)| \leq \frac{3\sqrt{3}}{8} < 1$ for all x, meaning that g is a contraction. Thus if x_* is the root of f, we have

$$|g(x) - x_*| \leq \frac{3\sqrt{3}}{8} \left| x - x_* \right|,$$

So $|x_n - x_*| \approx \left(\frac{3\sqrt{3}}{8}\right)^n$.

(b) In this case, g is no longer a contraction, but it is after a critical value of x. However, the derivative of g at the fixed point is seen to be much smaller than in part (a); it can be estimated at 0.18. So we get faster convergence. See Figure 1.

□

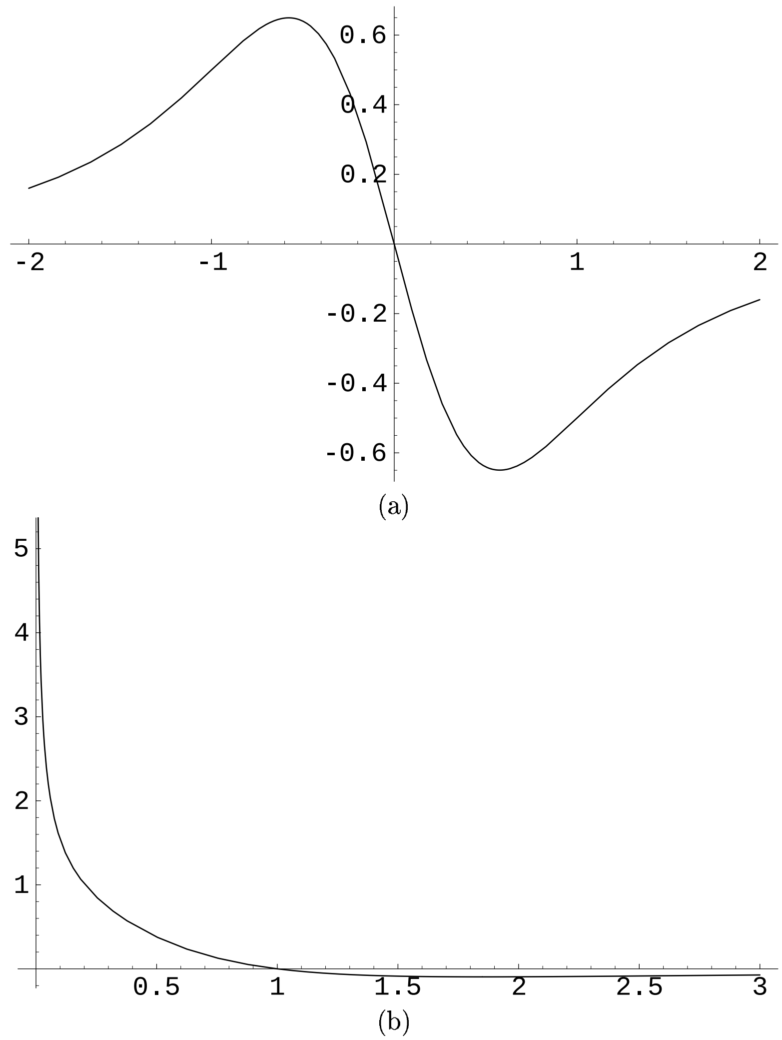

FIGURE 1. (a) $g'(x)$, where $g(x) = (1+x^2)^{-1}$. Note that $|g'(x)| <$ const. < 1, so g is a contraction. (b) $g'(x)$, where $g(x) = \big(x/(1+x^2)\big)^{1/2}$. This time g is not a contraction, but the derivative at the fixed point is much smaller in absolute value.