MATH 118 : SPRING 1999
SOLUTIONS TO PROBLEM SET 6

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All exercises are from [Dev92].

Exercise 9.5. *Find all points in Σ whose distance from $\mathbf{0} = (000\dots)$ is exactly $\frac{1}{2}$.*

Solution. Write $\Sigma = M_1 \cup M_{01} \cup M_{00}$, where the notation extends the definitions in Exercise 9.7. For $\mathbf{s} \in M_1$, $d[\mathbf{s}, \mathbf{0}] \geq 1$, so no point in M_1 is of a distance $\frac{1}{2}$ from $\mathbf{0}$. For $\mathbf{s} \in M_{01}$, $d[\mathbf{s}, \mathbf{0}] \geq \frac{1}{2}$, and in fact this distance is equal to $\frac{1}{2}$ if and only if $\mathbf{s} = (01\bar{0})$. Finally, for $\mathbf{s} \in M_{00}$, $d[\mathbf{s}, \mathbf{0}] \leq 1/2$, and the distance is exactly $\frac{1}{2}$ if and only if $\mathbf{s} = (00\bar{1})$. So there are exactly two points in Σ which are $\frac{1}{2}$ from $\mathbf{0}$. □

Exercise 9.7. *Let*

$$M_{01} = \{\, \mathbf{s} \in \Sigma \mid s_0 = 0,\ s_1 = 1 \,\}$$

and

$$M_{101} = \{\, \mathbf{s} \in \Sigma \mid s_0 = 1,\ s_1 = 0,\ s_2 = 1 \,\}.$$

What is the minimum distance between a point in M_{01} and a point in M_{101}? Give an example of two sequences that are this close to each other.

Solution. In order for a point of M_{01} to be close to a point of M_{101}, they must have as many digits as possible in common. So we restrict to the subset $M_{011} \subset M_{01}$. Then if $\mathbf{s} \in M_{011}$, write $\mathbf{s} = 011\mathbf{s}'$, and likewise for $\mathbf{t} \in M_{101}$. Then

$$\begin{aligned} d[\mathbf{s}, \mathbf{t}] &= d[011\mathbf{s}', 101\mathbf{t}'] \\ &= \frac{3}{2} + \frac{1}{8} d[\mathbf{s}', \mathbf{t}']. \end{aligned}$$

Since we can pick $\mathbf{s}' = \mathbf{t}' = \mathbf{0}$, we get that the minimum distance is $\frac{3}{2}$ (and provide an example at the same time). Incidentally, picking $\mathbf{s}' = \mathbf{0}$ and $\mathbf{t}' = \mathbf{1}$, we get the maximum distance is $\frac{7}{4}$. □

Exercise 9.10. *Let Σ_N denote the space of sequences whose entries are the positive integers $0, 1, \dots, N-1$, and let σ_N be the shift map on Σ_N. For $\mathbf{s}, \mathbf{t} \in \Sigma_N$, let*

$$d_N[\mathbf{s}, \mathbf{t}] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{N^i}.$$

Prove that d_N is a metric on Σ_N.

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Proof. Note that the sum in the definition of $d_N[\mathbf{s},\mathbf{t}]$ is a sum of nonnegative numbers, majorized by $\frac{N-1}{N^i}$, which sums to N. So the sum is absolutely convergent for all $\mathbf{s}$ and $\mathbf{t}$, and moreover $d_N[\mathbf{s},\mathbf{t}] \geq 0$. This shows that $\operatorname{diam}\Sigma_N = N$, and this distance is achieved by $\mathbf{0}$ and $\mathbf{N-1}$. This answers Exercise 9.11.

If $d_N[\mathbf{s},\mathbf{t}] = 0$, then we must have that $|s_i - t_i| = 0$ for all i, i.e, that $s_i = t_i$ for all i. So $\mathbf{s} = \mathbf{t}$. Therefore, we have established that d_N is positive-definite. That d_N is symmetric (that is, $d_N[\mathbf{s},\mathbf{t}] = d_N[\mathbf{t},\mathbf{s}]$ for all $\mathbf{s}$ and $\mathbf{t}$) is obvious.

Finally, we prove the triangle inequality. Let $\mathbf{s}$, $\mathbf{t}$, and $\mathbf{u}$ be given. Then

$$\begin{aligned} d_N[\mathbf{s},\mathbf{u}] &= \sum_{i=0^\infty} \frac{|s_i - u_i|}{N^i} \\ &= \sum_{i=0}^{\infty} \frac{|s_i - t_i + t_i - u_i|}{N^i} \\ &\leq \sum_{i=0}^{\infty} \left(\frac{|s_i - t_i|}{N^i} + \frac{|t_i - u_i|}{N^i} \right). \end{aligned}$$

Since the sum is absolutely convergent, we can rearrange the terms and write this sum as

$$\begin{aligned} &= \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{N^i} + \sum_{i=0}^{\infty} \frac{|t_i - u_i|}{N^i} \\ &= d_N[\mathbf{s},\mathbf{t}] + d_N[\mathbf{t},\mathbf{u}]. \end{aligned}$$

Hence d_N is a metric. □

Exercise 9.12. *How many fixed points does σ_N have? How many two-cycles? How many cycles of prime period two?*

Proof. Suppose $\sigma_N(\mathbf{s}) = \mathbf{s}$. Then $s_{i+1} = s_i$ for all $i \geq 0$, and thus $\mathbf{s} = (\bar{s}_0)$ for any choice of $s_0 \in \{0, \dots, N-1\}$. Hence there are N fixed points for σ.

If $\sigma_N^2(\mathbf{s}) = \mathbf{s}$, then $s_{i+2} = s_i$ for all $i \geq 0$, and thus $\mathbf{s} = (\overline{s_0 s_1})$ for any choice of $s_0, s_1 \in \{0, \dots, N-1\}$. Hence there are N^2 fixed points for σ_N^2. The choices where $s_0 = s_1$ are actually fixed points. So there are $N^2 - N$ points of prime period two. Since each two-cycle is composed of two points, there are $\frac{N^2-N}{2} = \binom{N}{2}$ cycles of prime period two. □

Exercise 9.13. *How many points in Σ_N are fixed by σ_N^k?*

Proof. This should be pretty easy by now. If $\sigma_N^k(\mathbf{s}) = \mathbf{s}$, then $s_{i+k} = s_i$ for all $i \geq 0$, so $\mathbf{s} = (\overline{s_0 \dots s_{k-1}})$. Hence there are N^k different such sequences. Breaking these up into cycles is much harder. □

Exercise 9.14. *Prove that $\sigma_N \colon \Sigma_N \to \Sigma_N$ is continuous.*

First, we extend the Proximity Theorem to N symbols.

Problem (Proximity Theorem′)**.** *Let $\mathbf{s},\mathbf{t} \in \Sigma$ and suupse $s_i = t_i$ for $i = 0, 1, \dots, k$. Then $d_N[\mathbf{s},\mathbf{t}] \leq N^{-k}$. Conversely, if $d_N[\mathbf{s},\mathbf{t}] < N^{-k}$, then $s_i = t_i$ for $i \leq k$.*

Proof. Suppose $s_i = t_i$ for $i \leq k$. Then

$$\begin{aligned} d_N[\mathbf{s}, \mathbf{t}] &= \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{N^i} \\ &= \sum_{i=k+1}^{\infty} \frac{|s_i - t_i|}{N^i} \\ &\leq \sum_{i=k+1}^{\infty} \frac{N-1}{N^i} \\ &= \frac{N-1}{N^{k+1}} \left(\frac{1}{1 - \frac{1}{N}} \right) \\ &= \frac{1}{N^k}. \end{aligned}$$

Conversely, suppose that $s_j \neq t_j$ for some $j \leq k$. Then

$$d_N[\mathbf{s}, \mathbf{t}] \geq \frac{1}{N^j} \geq \frac{1}{N^k},$$

and the contrapositive is what we want. □

Proof that σ_N is continuous. This is now completely easy. Given $\varepsilon > 0$, choose k such that $N^{-k} < \varepsilon$. Let $\delta = N^{-k-1}$. Then if $d_N[\mathbf{s}, \mathbf{t}] < \delta$, we have $s_i = t_i$ for $i \leq k+1$ by the Proximity Theorem. We may say later on that "the first $k+1$ digits of $\mathbf{s}$ and $\mathbf{t}$ agree" and this is what we mean. Technically, they agree on the first $k+2$ digits, but we include zero and do not count it. This is what mathematicicans call an *abuse of language.*

With this abuse in hand, we frolic down the primrose path of the rest of the proof. If the first $k+1$ digits of $\mathbf{s}$ and $\mathbf{t}$ agree, then the first k digits of $\sigma_N(\mathbf{s})$ and $\sigma_N(\mathbf{t})$ agree, which implies that

$$d_N[\sigma_N(\mathbf{s}), \sigma_N(\mathbf{t})] \leq N^{-k} < \varepsilon.$$

So σ_N is (uniformly, in fact!) continuous. □

Exercise 9.18. *Each of the following defines a function on the space of sequences Σ. In each case, prove whether the given function is continuous. (Pick any three.)*

a. $F(s_0 s_1 s_2 \dots) = (0 s_0 s_1 s_2 \dots)$
b. $G(s_0 s_1 s_2 \dots) = (0 s_0 0 s_1 0 s_2 \dots)$
c. $H(s_0 s_1 s_2 \dots) = (s_1 s_0 s_3 s_2 s_5 s_4 \dots)$
d. $J(s_0 s_1 s_2 \dots) = (\hat{s}_0 \hat{s}_1 \hat{s}_2 \dots)$ *where* $\hat{s}_j = 1$ *if* $s_j = 0$ *and* $\hat{s}_j = 0$ *if* $s_j = 1$.
e. $K(s_0 s_1 s_2 \dots) = \big((1-s_0)(1-s_1)(1-s_2)\dots\big)$
f. $L(s_0 s_1 s_2 \dots) = (s_0 s_2 s_4 s_6 \dots)$
g. $M(s_0 s_1 s_2 \dots) = (s_1 s_{10} s_{100} s_{1000} \dots)$
h. $N(s_0 s_1 s_2 \dots) = (t_0 t_1 t_2 \dots)$, *where* $t_j = s_0 + s_1 + \cdots + s_j \bmod 2$. *That is,* $t_j = 0$ *if* $s_0 + \cdots + s_j$ *is even, and* $t_j = 1$ *if* $s_0 + \cdots + s_j$ *is odd.*
i. $P(s_0 s_1 s_2 \dots) = (t_0 t_1 t_2 \dots)$, *where* $t_j = \lim_{n \to \infty} s_n$ *if this limit exists, and* $t_j = s_j$ *otherwise.*

Proofs. a. Consider that

$$d_N[0\mathbf{s}, 0\mathbf{t}] = \frac{1}{2} d[\mathbf{s}, \mathbf{t}].$$

So d_N is continuous, indeed, a contraction mapping. What do you think the unique fixed point is?

b. Given $\varepsilon > 0$, choose n such that $2^{-2n} < \varepsilon$. Let $\delta = 2^{-n}$. Then if $d[\mathbf{s}, \mathbf{t}] < \delta$, the first n digits of $\mathbf{s}$ and $\mathbf{t}$ agree, so the first $2n$ digits of $G(\mathbf{s})$ and $G(\mathbf{t})$ agree, so $d[G(\mathbf{s}), G(\mathbf{t})] \leq 2^{-2n} < \varepsilon$. So σ is continuous.

c. Given $\varepsilon > 0$, choose n such that $2^{-n} < \varepsilon$ and n is odd. Then let $\delta = 2^{-n}$. If $d[\mathbf{s}, \mathbf{t}] < \delta$, then the first n digits of $\mathbf{s}$ and $\mathbf{t}$ agree, so the first n digits of $H(\mathbf{s})$ and $H(\mathbf{t})$ agree, too, so $d[H(\mathbf{s}), H(\mathbf{t})] \leq 2^{-n} < \varepsilon$. Therefore, H is continuous.

d. Given $\varepsilon > 0$, choose n such that $2^{-n} < \varepsilon$. Then let $\delta = 2^{-n}$. If $d[\mathbf{s}, \mathbf{t}] < \delta$, then $s_i = t_i$ for $i \leq n$. This is true if and only if $\hat{s}_i = \hat{t}_i$ for $i \leq n$, so the first n digits of $J(\mathbf{s})$ and $J(\mathbf{t})$ agree. Therefore, $d[J(\mathbf{s}), J(\mathbf{t})] \leq 2^{-n} < \varepsilon$, and J is continuous.

e. Note that $K = J$. So K is continuous by a previous exercise.

f. Given $\varepsilon > 0$, choose k such that $2^{-k} < \varepsilon$. Then let $\delta = 2^{-10^k}$. Then if $d[\mathbf{s}, \mathbf{t}] < \delta$, the first 10^k digits of $\mathbf{s}$ and $\mathbf{t}$ agree, so the first k digits of $M(\mathbf{s})$ and $M(\mathbf{t})$ agree, so $d[M(\mathbf{s}), M(\mathbf{t})] \leq 2^{-k} < \varepsilon$. Thus M is continuous.

g. Given $\varepsilon > 0$, choose n such that $2^{-n} < \varepsilon$. Then let $\delta = 2^{-n}$. If $d[\mathbf{s}, \mathbf{t}] < \delta$, then $s_i = t_i$ for $i \leq n$, so certainly for all $j \leq n$, the sums $\sum_{i=0}^{j} s_i$ and $\sum_{i=0}^{j} t_i$ are equal. The equality remains after reduction *modulo* two. So $N(\mathbf{s})$ and $N(\mathbf{t})$ agree on their first n digits, and thus $d[N(\mathbf{s}), N(\mathbf{t})] \leq 2^{-n} < \varepsilon$.

h. We claim that P is not continuous. To prove this, we recall what it means to not be continous. The sentence "P is continous at $\mathbf{s} \in \Sigma$" means that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $\mathbf{t} \in \Sigma$, if $d[\mathbf{s}, \mathbf{t}] < \delta$, then $d[P(\mathbf{s}), P(\mathbf{t})] < \varepsilon$. The negation of this proposition is that *there exists* an $\varepsilon > 0$ such that *for all* $\delta > 0$ *there exists* $\mathbf{s} \in \Sigma$ such that $d[\mathbf{s}, \mathbf{t}] < \delta$ *but* $d[P(\mathbf{s}), P(\mathbf{t})] \geq \varepsilon$.

Let $\mathbf{s} = \mathbf{0}$ and $\varepsilon = 1$. Then given $\delta > 0$, choose n such that $2^{-n} < \delta$. Let

$$\mathbf{t} = (\underbrace{00\ldots0}_{n+1}11\ldots)$$

Then $d[\mathbf{0}, \mathbf{t}] \leq 2^{-n}$, but

$$d[P(\mathbf{0}, P(\mathbf{t})] = d[\mathbf{0}, \mathbf{1}] = 2 > 1 = \varepsilon.$$

So P is not continuous at $\mathbf{0}$.

The moral of the story is that many many functions on a totally disconnected space can be continuous!

Since no solution set would be complete without a graphic, I made a histogram of the parts of this problem attempted by students. It's a trivial use of the `Statistics‘DataManipulation‘` and `Graphics‘Graphics‘` packages. We observe that many students are not so adventurous. □

References

[Dev92] Robert L. Devaney, *A First Course in Chaotic Dynamical Systems: Theory and Experiment*, Addison-Wesley, 1992.

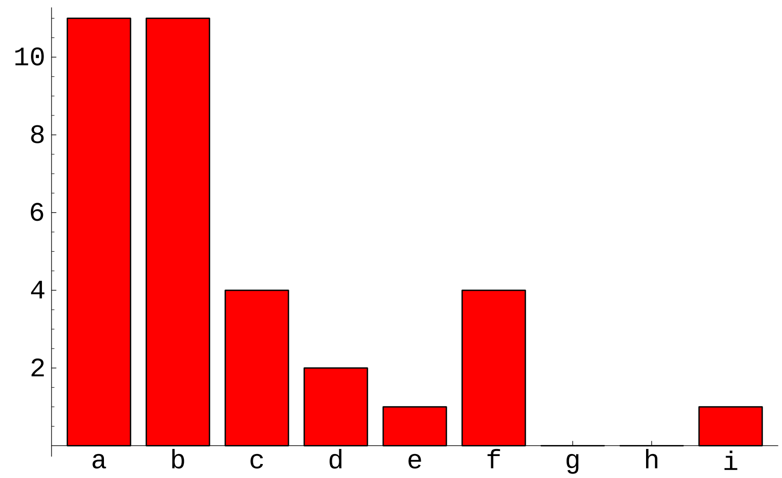

FIGURE 1. Histogram of parts of Exercise 9.18 attempted by students.