Math 119 Partial differential equations and Applications

Take Home Midterm: October 27, 2005 - November 3, 2005 *Robert Strain*

Problem 1 A function $u : \bar{\Omega} \to \mathbb{R}$ is said to be *subharmonic* if

$$-\Delta u \leq 0 \text{ in } \Omega.$$

If $\Omega \subset \mathbb{R}^3$ is an open connected set, prove the following:

(a) u satisfies the maximum principle: $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.

(b) For every open ball $B(x,r) \subset \Omega$, u satisfies the following mean value inequality

$$u(x) \leq \frac{1}{4\pi r^2} \int_{\partial B(x,r)} u(y) dS(y).$$

(c) Suppose $v : \bar{\Omega} \to \mathbb{R}$. Say $B(x,r) \subset \Omega$ and

$$\begin{aligned} \Delta v &= 0 \text{ in } B(x,r), \\ v &= u \text{ on } \partial B(x,r), \end{aligned}$$

Prove $u \leq v$ in $B(x,r)$.

This is a *comparison theorem*; similar theorems for other partial differential equations.

Problem 2 Let $u(x,t)$ solve the wave equation on $\mathbb{R}^3$,

$$\Box u = u_{tt} - c^2 \Delta u = 0,$$

with smooth initial conditions $u(x,0) = 0$, $u_t(x,0) = h(x)$. Suppose that h is zero outside the ball $B(0,\epsilon)$ of radius ϵ about the origin, and $|h(x)| \leq M$ for all x. Show that, for $t > 0$,

$$|u(x,t)| \leq M\epsilon/(c^2 t).$$

Problem 3

(a) Show that in 3 dimensions the general solution of the wave equation $\Box u = 0$ with spherical symmetry about the origin, i.e. $u(x,t) = u(r,t)$ with $r = |x|$, has the form

$$u(r,t) = \frac{F(r+ct) + G(r-ct)}{r}.$$

for suitable F, G.

(b) Show that the solution with initial data of the form

$$u(x,0) = 0, \quad u_t(x,0) = g(r)$$

(g an even function of r) is given by

$$u(r,t) = \frac{1}{2cr}\int_{r-ct}^{r+ct} \rho g(\rho) d\rho. \tag{1}$$

(c) For

$$g(r) = \begin{cases} 1 & \text{for } 0 < r < a, \\ 0 & \text{for } r > a. \end{cases}$$

find u explicitly from (1) in the different regions bounded by the cones $r = a \pm ct$ in (x,t)-space. Show that u is discontinuous at $(0, a/c)$, (due to the focussing of the discontinuity of u_t at $t = 0$, $|x| = a$).

Problem 4 The Landau equation from plasma physics can be written as

$$\frac{\partial f(t,v)}{\partial t} = \sum_{i,j=1}^{3} \frac{\partial}{\partial v_j} \int_{\mathbb{R}^3} b_{ij}(v - v^*) \left\{ f(t,v^*) \frac{\partial f(t,v)}{\partial v_i} - \frac{\partial f(t,v^*)}{\partial v_i^*} f(t,v) \right\} dv^*. \tag{2}$$

Above the 3×3 matrix $\mathbf{b} = (b_{ij})$ is given by

$$b_{ij}(v) = \frac{1}{|v|}\left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right).$$

The Landau equation can be re-written in the form

$$\frac{\partial f}{\partial t} = \sum_{i,j=1}^{3} a_{ij}(f) \frac{\partial^2 f}{\partial v_j \partial v_i} + cf^2. \tag{3}$$

(a) Show that $\sum_{i,j=1}^{3} \frac{\partial^2 b_{ij}(v-v^*)}{\partial v_j \partial v_i^*} = 0$.

(b) Assume the solution to the first form of the Landau equation (2) is sufficiently smooth. Derive the second form of the Landau equation (3) from (2).

(c) What is the matrix $a_{ij}(f(t,v))$?

(d) What is the constant c?

[Although currently the existence of smooth solutions to this Landau equation is unknown, the second equation is believed to be easier to work with. And a proof of existence of smooth solutions to the second form would imply existence of smooth solutions to the first form.]