as $q = b_0 p_0 + \cdots + b_n p_n$ for some $b_0, \ldots, b_n \in \mathbb{F}$

Since we want $q(c_0) = a_0$, we need

$$b_0 p_0(c_0) + \cdots + b_n p_n(c_0) = a_0$$

$$\Rightarrow \quad b_0 . 1 + b_1 . 0 + \cdots + b_n . 0 = a_0$$

$$\Rightarrow \quad b_0 = a_0$$

Similarly, we need $b_1 = a_1, \ldots, b_n = a_n$.

So if q exists, it must be

$$q = a_0 p_0 + \cdots + a_n p_n$$

Check: does this work? Well, $q(c_0) = a_0 . 1 + a_1 . 0 + \cdots + a_n . 0 = a_0$

and $q(c_1) = a_1$ and ... and $q(c_n) = a_n$, so yes: this works.

Thus such a polynomial exists and is unique:

$$q(x) = \sum_{i=0}^{i=n} a_i p_i(x)$$

(d) Apply part (c) with $a_0 = p(c_0), \ldots, a_n = p(c_n)$. There is a unique polynomial $q(x) \in V$ with $q(c_i) = p(c_i)$, given by $q(x) = \sum_{i=0}^{i=n} p(c_i) p_i(x)$. But $p(c_i) = p(c_i)$ (duh) and $p \in V$, so (by uniqueness) $p(x) = \sum_{i=0}^{i=n} p(c_i) p_i(x)$

(e) We know $p(x) = \sum_{i=0}^{i=n} p(c_i) p_i(x)$. Integrate this!