$\boxed{2.3.11}$ $T: V \to V$ linear. Then $T^2 = T_0 \iff R(T) \subseteq N(T)$.

"$\Longrightarrow$" Suppose $T^2 = T_0$

Let $w \in R(T)$. Then $w = T(v)$ for some $v \in V$.

But: $T(T(v)) = T^2(v) = 0 \Rightarrow T(w) = 0 \Rightarrow w \in N(T)$

Hence $R(T) \subset N(T)$.

"$\Longleftarrow$" Suppose $R(T) \subseteq N(T)$

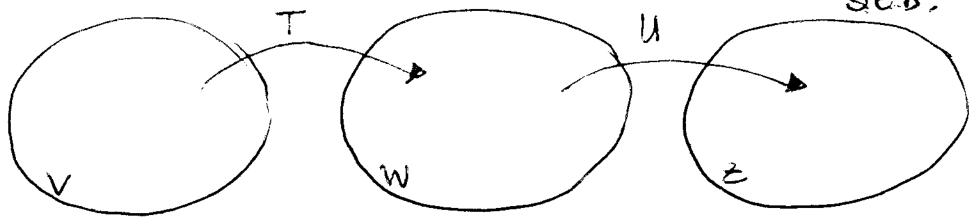
Look at $T(T(v))$ for some arbitrary $v \in V$.

$T(\underbrace{T(v)}_{=w \in V}) = \underbrace{T(w)}_{=w' \in R(T)} = w'$ $\Big|$ $\Rightarrow w' \in N(T) \Rightarrow w' = 0$, $T(w) = 0$, and therefore $T^2(v) = 0$.

Since v was chosen arbitrarily, $T^2 = T_0$ for all $v \in V$.

QED.

$\boxed{2.3.12}$



(a) UT is one-to-one $\Rightarrow$ T is one-to-one.

Prove by contradiction. Suppose $\exists\, v_1 \neq v_2 \in V$ such that $T(v_1) = T(v_2) = w$

Then $UT(v_1) = U(w) = UT(v_2)$ – but this is a contradiction, since U is one-to-one

U need not be one-to-one. Can you think of an example?

(b) UT is onto $\Rightarrow$ U is onto.

Know: $\forall z \in Z$, $\exists\, v \in V$ such that $UT(v) = z$.

but, by defn, $T(v) = w \in W$. Hence $U(w) = z$, and therefore any element in Z can be written as $U(w)$ for some $w \in W$.

Again, T need not be onto. Why?

(c) $U, T \begin{cases}\text{one-to-one} \\ \text{onto}\end{cases} \Rightarrow UT$ is $\begin{cases}\text{one-to-one} \\ \text{onto}\end{cases}$

To shorten the proof, just show: $R(UT) = Z$ and $N(UT) = \{0_V\}$

(i) $UT(v) = U(T(v)) = U(w)$ for $w \in W$. ~~Since~~

Let $z \in Z$. Since U is onto, $\exists\, w \in W$ such that $U(w) = z$. $\Big\}$ $\Rightarrow$

Since T is onto, $\exists\, v \in V$ such that $T(v) = w$.

$\Rightarrow \exists\, v \in V$ such that $UT(v) = z$ for any $z \in Z$. Hence $R(UT) = Z$.

(ii) $N(UT) = 0$. Suppose $v \in V$ such that $UT(v) = 0_Z$. $\Rightarrow T(v) = w \in W$.

Since U is one-to-one, $N(U) = \{0_W\} \Rightarrow T(v) = w = 0_W$.

Since T is one-to-one, $N(T) = \{0_V\} \Rightarrow \boxed{v = 0_V}$, Hence $N(UT) = \{0_V\}$.

This completes the proof. 4