# 1 Section 2.1

## 1.1 Problem 40

### 1.1.1 (a)

$\eta$ is clearly onto $V/W$ because for any $v + W \in V/W$, $\eta(v) = v + W$. If $v \in N(\eta)$, $v + W = 0 + W \Rightarrow v - 0 \in W$ i.e. $vinW$, and $\eta(v) = W = 0 + W$ for all $v \in W$. So $n(\eta) = W$. Finally $\eta(av + bu) = av + bu + W = a(v + W) + b(u + W)$ (by 1.3 Ex. 31) and so $\eta(av + bu) = a\eta(v) + b\eta(u)$, i.e. $\eta$ is linear.

### 1.1.2 (b)

Since $R(\eta) = V/W$ and $N(\eta) = W$, the dimension theorem tells us dim $V$ = dim $W$+ dim $V/W$.

### 1.1.3 (c)

The proof of 1.6 ex. 35 uses the same method as the proof of the dimension theorem (i.e. constructing a basis) whereas (b) just applies the result of the dimension theorem.

# 2 Section 2.4

## 2.1 Problem 24

### 2.1.1 (a)

From 1.3 Ex. 31 and 2.1 Ex. 40, $v + N(T) = v' + N(T) \Rightarrow v - v' \in N(T) \Rightarrow T(v - v') = 0 \Rightarrow T(v) = T(v')$ by linearity.

### 2.1.2 (b)

$\bar{T}(a(v + N(T)) + b(u + N(T))) = \bar{T}(av + bu + N(T)) = T(av + bu) = aT(v) + bT(u) = a\bar{T}(v + N(T)) + b\bar{T}(u + N(T))$. The first equality follows from 1.3 Ex. 31.

### 2.1.3 (c)

$N(\bar{T}) = \{v + N(T)|T(v) = 0\}$ . Since the only such $v$ are $v \in N(T)$, $N(\bar{T}) = \{v + N(T)|v \in N(T)\} = \{0 + N(T)\}$ which says exactly that $T$ is one-to-one. Since $T$ is onto $Z$, any vector in $Z$ is of the form $T(v)$ and so $\bar{T}(v + N(T)) = T(v)$ shows $\bar{T}$ is onto. So $\bar{T}$ is onto and one-to-one which proves it is an isomorphism.

### 2.1.4 (d)

$\bar{T}(\eta(v)) = \bar{T}(v + N(T)) = T(v) \Rightarrow T = \bar{T}\eta$.

(a)-(d) are collectively known as the *First Isomorphism Theorem* for vector spaces. The Theorem is also true for many more general objects such as groups, rings, and algebras.

# 3 Section 2.7

## 3.1 Problem 2

### 3.1.1 (a)

False. Only subspaces of the form given by Theorem 2.34 are the solution space of such an equation. The subspace generated by $\{x^2\}$, for example, is not such a subspace.

### 3.1.2 (b)

False. For $te^{ct}$ to be a solution, $e^{ct}$ must be also by 2.34. In this case $(c = 0)$ the latter is equal to 1, which is not in the solution set.

### 3.1.3 (c)

True. If $p(D)x = 0$ then by diffentiating $p(D)x' = 0$.

### 3.1.4 (d)

True. $p(D)q(D)(x + y) = p(D)q(D)(x) + p(D)q(D)(y) = p(D)q(D)(x) + p(D)(0) = p(D)q(D)(x)$. Since $p(D)(x) = 0$ we have by (c) above and taking linear combinations that $p(D)(q(D)(x)) = 0$.

### 3.1.5 (e)

False. Let $p(t) = t - 1$ and $q(t) = t - 2$, $x = e^t$ and $y = e^{2t}$. Then $xy = e^{3t}$ but since 3 is not a root of $p(t)q(t) = (t - 1)(t - 2)$ it is not in the nullspace of $p(D)q(D)$.

## 3.2 Problem 12

$0 = p(D)(V) = h(D)g(D)(V) \Rightarrow g(D)(V) \subset N(h(D))$ and by definition $R(g(D_V)) = g(D)(V)$. From previous exercises we know $n = \dim\ V = \dim\ N(p(D)) = \dim\ N(h(D)) + \dim\ N(g(D))$, and by the dimension theorem $n = \dim V = \dim N(g(D)) + \dim\ R(g(D))$ so $\dim\ R(g(D)) = \dim\ N(h(D))$. As suggested by the hint, this completes the proof.

### 3.3 Problem 13

#### 3.3.1 (a)

Since $(D - cI)$ is onto $C^\infty$ for any complex $c$ (Lemma 1) we have by induction that any differential operator $p(D)$ is also. This proves that for some $y$, $p(D)(y) = x$.

#### 3.3.2 (b)

Certainly $z + y$ is a solution for any $y \in V$. Now assume there is some $w$ such that $p(D)(w) = x$. Then $p(D)(w) = p(D)(z)$ for some fixed $z$ in the solution set and so $p(D)(w) - p(D)(z) = 0 \Rightarrow w - z = y$ where $y$ is in the nullspace of the homogenous equation, so for the general solution $w$ we have $w = z + y$

### 3.4 Problem 18

#### 3.4.1 (a)

It is easy to check (using the quadratic formula) that the auxiliary polynomial has roots $c_1 = -r/(2m) + sqrt((r/2m)^2 - (k/m))$ and

$c_2 = -r/(2m) - sqrt((r/2m)^2 - (k/m))$ so the basis for the space of solutions is give by: $\{e^{c_1 t}, e^{c_2 t}\}$ if $((r/2m)^2 - (k/m)) \neq 0$ and $\{e^{c_1 t}, te^{c_1 t}\}$ otherwise.

#### 3.4.2 (b)

For simplicity we only check the case $(r/2m)^2 = k/m$ (the other case is similar). Assume $y(t) = Ce^{c_1 y} + Dte^{c_1 t}$. Then $y(0) = 0 \Rightarrow C = 0$ and $y'(0) = v_0 \Rightarrow D = v_0$ so $y(t) = v_0 te^{-rt/2m}$.

## 4 Section 3.2

### 4.1 Problem 6

#### 4.1.1 (d)

With respect to the standard bases for $\mathbb{R}^3$ and $P_2(\mathbb{R})$, we have

$$(T) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The easiest way to find the inverse is to simultaneously row-reduce this and the identity matrix; this gives, again with respect to the standard basis,

$$(T^{-1}) = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}$$

so in terms of polynomials, $T^{-1}(a_0 + a_1 x + a_2 x^2 = (a_2, \frac{1}{2}(a_0 - a_1), \frac{1}{2}(a_0 + a_1) - a_2)$.

### 4.1.2 (e)

The method of computing the inverse is the same as in (d). The matrix of the inverse is:

$$(T^{-1}) = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

## 4.2 Problem 17

If B is 3 x 1, we know that the image of $B$ as a linear transformation $F^3 \to F^1$ has dimension at most 1, so by the dimension theorem the nullspace of $B$ has dimension at least 2. From Section 2.7 $N(BC) = N(B) + N(C)$ if $C$ is onto, in which case $N(BC) \geq 2$. The only other possibility is $C = 0$ and then $N(BC) = 3 \geq 2$ so we know that the rank of $BC$ is at most 1 in any case. If $A$ is 3 x 3 with columns $(A_1, A_2, A_3)$ and has rank 1, we know that the column rank of of $A$ is 1 so the $A_i$ are all multiples of a common vector $v$, say $A_i = c_i v$. Then if we consider $(v)$ as a 3 x 1 matrix, we have $A = (v)(c_1, c_2, c_3)$.

# 5 Section 3.3

## 5.1 Problem 3

For each of these systems, write down the matrix of coefficients and then compute the reduced row echelon form.

### 5.1.1 (d)

The row echelon form is:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which corresponds to $x_1 = 2$, $x_2 - x_3 = 1$ so the solutions are of the form $\{(2, 1, 0)^t + s(0, 1, 1)^t | s \in F\}$.

### 5.1.2 (g)

The row echelon form is:

$$\begin{bmatrix} 1 & 0 & 3 & -1 & -1 \\ 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

which corresponds to $x_1 + 3x_3 - x_4 = -1$, $x_2 - x_3 + x_4 = 1$ so the solutions are of the form $\{(-1, 1, 0, 0)^t + s(-3, 1, 1, 0)^t + r(1, -1, 0, 1) | s, r \in F\}$.

## 5.2 Problem 10

Let $Ax = b$ where $A$ is the coefficient matrix of a system of $m$ linear equations in $n$ unknowns, and $A$ has rank $m$. So the columns of $A$ span a subspace of dimension $m$; since $b \in F^m$, this means the columns of $A$ together with $b$ form a linearly dependent set (since there are at least $m + 1$ of these vectors) and so rank $(A) =$ rank $(A|b) \Rightarrow$ the system is consistent by Theorem 3.11.