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(a) We need to show $(R+\lambda S)^*(x) = R^*(x) + \bar{\lambda} S^*(x)$ for all $x \in V$, so it is enough to show that

$$\langle (R+\lambda S)^*(x), y\rangle = \langle R^*(x) + \bar{\lambda} S^*(x), y\rangle$$

for all $x, y \in V$

But

$$\begin{aligned}\langle (R+\lambda S)^*(x), y\rangle &= \langle x, R(y) + \lambda S(y)\rangle \\ &= \langle x, R(y)\rangle + \bar{\lambda}\langle x, S(y)\rangle \\ &= \langle R^*(x), y\rangle + \bar{\lambda}\langle S^*(x), y\rangle \\ &= \langle R^*(x) + \bar{\lambda} S^*(x), y\rangle,\end{aligned}$$

for any $x, y \in V$.

(b)

$$\begin{aligned}\|T(x)\|^2 = \langle Tx, Tx\rangle &= \langle x, T^*Tx\rangle \\ &= \langle x, TT^*x\rangle \\ &= \langle T^*x, T^*x\rangle \\ &= \|T^*x\|^2\end{aligned}$$

Since $\|Tx\|$ and $\|T^*x\|$ are both non-negative, we see that $\|Tx\| = \|T^*x\|$.

(c) It is enough to show (see the hint) that $\|(T-\lambda I)(x)\| = 0 \iff \|(T^* - \bar{\lambda} I)(x)\| = 0$

But by (a) and (b), $\|(T-\lambda I)(x)\| = \|(T^*-\bar{\lambda} I)(x)\|$.