Consider the generating function

$$p(t) = (\lambda_1 + \dots + \lambda_n) + (\lambda_1^2 + \dots + \lambda_n^2)t + \dots$$

$$= \sum_{i=1}^{i=n} \sum_{r \geq 1} \lambda_i^r t^{r-1}$$

$$= \sum_{i=1}^{i=n} \frac{\lambda_i}{1 - \lambda_i t}$$

Then if $h(t) = \prod_{i=1}^{i=n} \frac{1}{1-\lambda_i t}$

then $d/dt\,(\log h(t)) = -\sum_{i=1}^{i=n} \frac{\lambda_i}{1-\lambda_i t}$

and so $p(t) = -d/dt \log h(t)$

If $\lambda_1 + \dots + \lambda_n = 0$ and $\lambda_1^2 + \dots + \lambda_n^2 = 0$ and ...

then $p(t) = 0 \Rightarrow \log h(t) = \text{constant}$ (independent of t)

$\Rightarrow h(t)$ is independent of t

But $h(0) = 1$ [regardless of the value of the λ_i] and

so $h(t) = 1$.

Thus $\frac{1}{h(t)} = 1$ too, so $(1-\lambda_1 t) \dots (1-\lambda_n t) = 1$

$\Rightarrow 1 - (\lambda_1 + \dots + \lambda_n)t + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \dots)t^2 - \dots = 1$

$\Rightarrow \lambda_1 + \dots + \lambda_n = 0$

$\sum_{i<j} \lambda_i \lambda_j = 0$

etc. $\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0$