Thus $\{T(v_{k+1}), \dots, T(v_{k+l})\}$ is a basis for $R(T)$.

Putting everything together, we see that

$$\text{nullity}(T) = \dim N(T) = k$$

$$\text{rank}(T) = \dim R(T) = l$$

and $\dim(V) = k+l$ as $\{v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}\}$ is a basis for V.

Thus $\text{rank}(T) + \text{nullity}(T) = \dim V$. $\square$

(b) To show that W_0 is a subspace, I need to show that

- $0 \in W_0$
- if f and $g \in W_0$ then $f+g \in W_0$
- if $f \in W_0$ and $c \in \mathbb{R}$ then $cf \in W_0$

But the zero polynomial 0 evaluated at $1 \in \mathbb{R}$ gives zero, so $0 \in W_0$. Also, if f and g are in W_0 then $f(1) = 0$ and $g(1) = 0$, so $(f+g)(1) = f(1) + g(1) = 0$ also, and so $f+g \in W_0$.

Similarly, if $f \in W_0$ and $c \in \mathbb{R}$ then $(cf)(1) = cf(1) = c.0 = 0$

so $cf \in W_0$ too. Thus W_0 is a subspace.