It remains to show that β spans $N(T\mathcal{U})$. But if $w \in N(T\mathcal{U})$ then $\mathcal{U}(w) \in N(T)$ and so

$$\mathcal{U}(w) = a_1 y_1 + \cdots + a_r y_r \qquad \text{for some } a_1, \ldots, a_r \in \mathbb{R}$$

Thus
$$\mathcal{U}(w) = \mathcal{U}(a_1 v_1 + \cdots + a_r v_r)$$

$$\Rightarrow \quad w - a_1 v_1 - \cdots - a_r v_r \in N(\mathcal{U})$$

$$\Rightarrow \quad w - a_1 v_1 - \cdots - a_r v_r = b_1 x_1 + \cdots + b_s x_s$$
for some $b_1, \ldots, b_s \in \mathbb{R}$

$$\Rightarrow \quad w = a_1 v_1 + \cdots + a_r v_r + b_1 x_1 + \cdots + b_s x_s$$

Thus β spans $N(T\mathcal{U})$, and hence is a basis for $N(T\mathcal{U})$. $\square$

<u>Claim</u>: $\dim N(T\mathcal{U}) = \dim N(T) + \dim N(\mathcal{U})$

<u>Proof</u> Duh:

$$\dim N(T\mathcal{U}) = r + s$$
$$\dim N(T) = r$$
$$\dim N(\mathcal{U}) = s$$
$\square$

<u>Claim</u>: The sol^n space to $y'' - 4y = 0$ is 2-dimensional

<u>Proof</u> The solution space is $N(D^2 - 4I)$, where $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is differentiation.
But $D^2 - 4I = (D + 2I)(D - 2I)$
and we know from class that $D - 2I$ is onto and has a 1-dimensional null space, and that $D + 2I$ has a 1-dimensional null-space.
Putting $\mathcal{U} = D - 2I$, $T = D + 2I$ and using the