Q1 (a) $\operatorname{tr}(A) = \sum_{i=1}^{i=n} A_{ii}$

so $\operatorname{tr}(AB) = \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} A_{ij} B_{ji}$

$= \sum_{j=1}^{j=n} \sum_{i=1}^{i=n} B_{ji} A_{ij}$

$= \operatorname{tr}(BA)$

Matrices C and D are similar $\Leftrightarrow$ $C = Q^{-1}DQ$ for some invertible matrix Q

so if C and D are similar then $\operatorname{tr}(C) = \operatorname{tr}(Q^{-1}DQ)$
$= \operatorname{tr}(DQQ^{-1})$
$= \operatorname{tr}(D)$

(b) Put
$$D(x_1, \dots, x_n) = \begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & & & \\ 1 & x_n & \cdots & x_n^{n-1} \end{vmatrix}$$

If $x_i = x_j$ for any $i \neq j$ then the determinant will vanish (as two of the rows will become equal)

so D is divisible by $\prod_{i>j} (x_i - x_j)$

If we let each x_i have degree 1 then $D(x_1, \dots, x_n)$ has degree $0 + 1 + \dots + n-1 = \frac{1}{2}n(n-1)$

every entry in the first column has degree zero

every entry in the second col. has degree 1

every entry in the last column has degree $n-1$

and in expanding the determinant we take exactly one entry from each column

But the degree of $\prod_{i>j} (x_i - x_j)$ is also $\frac{1}{2}n(n-1)$
[it is the product of $\frac{1}{2}n(n-1)$ terms, each of which has

degree 1. So $D = C\prod_{i>j}(x_i - x_j)$

where C has degree zero, i.e. is a constant.

We need to show this constant is equal to 1.

<u>Base case</u>: $n=2$

$$D(x_1, x_2) = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = x_2 - x_1$$

$$= 1.(x_2 - x_1) \qquad \text{so } C = 1 \checkmark$$

<u>Induction step</u>: We know that $D(x_1,\dots,x_n) = C\prod_{i>j}(x_i - x_j)$

and we are assuming that $D(x_1,\dots,x_{n-1}) = \prod_{\substack{i>j \\ i\le n-1}}(x_i - x_j)$

We want to show that $C = 1$.

Put $x_1 = 0$. Then $D(0, x_2, \dots, x_n) = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 1 & x_2 & \dots & x_2^{n-1} \\ \vdots & & & \\ 1 & x_n & \dots & x_n^{n-1} \end{vmatrix}$

$$= \begin{vmatrix} x_2 & \dots & x_2^{n-1} \\ \vdots & & \vdots \\ x_n & \dots & x_n^{n-1} \end{vmatrix}$$

$$= x_2 x_3 \dots x_n \begin{vmatrix} 1 & x_2 & \dots & x_2^{n-1} \\ 1 & x_3 & \dots & x_3^{n-1} \\ \vdots & & & \\ 1 & x_n & \dots & x_n^{n-1} \end{vmatrix} = x_2 \dots x_n \prod_{\substack{i>j \\ i\le n \\ j\ge 2}}(x_i - x_j)$$

by the induction hypothesis

On the other hand, putting $x_1 = 0$ turns

$$C\prod_{i>j}(x_i - x_j) \quad \text{into} \quad C(x_2 - 0)\dots(x_n - 0)\prod_{\substack{i>j \\ j\ne 1}}(x_i - x_j)$$

$$= C x_2 \dots x_n \prod_{\substack{i>j \\ j\ge 2}}(x_i - x_j)$$

So $C = 1$. By induction on n we are done.

(c) First let's compute $T^*: P_1(\mathbb{R}) \to P_2(\mathbb{R})$.

This will be easier if we find o.n. bases for $P_2(\mathbb{R})$ and $P_1(\mathbb{R})$ [these <u>won't</u> be the o.n. bases which we are looking for].

Now
$$\langle 1, 1\rangle = 2 \qquad \langle 1, x\rangle = 0 \qquad \langle 1, x^2\rangle = 2/3$$
$$\langle x, 1\rangle = 0 \qquad \langle x, x\rangle = 2/3 \qquad \langle x, x^2\rangle = 0$$
$$\langle x^2, 1\rangle = 2/3 \qquad \langle x^2, x\rangle = 0 \qquad \langle x^2, x^2\rangle = 2/5$$

So applying Gram-Schmidt to the standard basis for $P_2(\mathbb{R})$ gives an o.n. basis $\tilde{\beta} = \{y_1, y_2, y_3\}$ for $P_2(\mathbb{R})$, where:

$$y_1 = \frac{1}{\sqrt{2}}$$

$$y_2 = \frac{1}{\text{length}}\left(x - \underbrace{\langle x, y_1\rangle}_{\text{zero}} y_1\right)$$

$$= \sqrt{\tfrac{3}{2}}\, x$$

$$y_3 = \frac{1}{\text{length}}\left(x^2 - \langle x^2, y_2\rangle y_2 - \langle x^2, y_1\rangle y_1\right)$$

$$= \frac{1}{\text{length}}\left(x^2 - \sqrt{2}/3\right)$$

$$= \frac{\sqrt{45}}{\sqrt{38 - 20\sqrt{2}}}\left(x^2 - \sqrt{2}/3\right)$$

Similarly, an o.n. basis $\tilde{\gamma}$ for $P_1(\mathbb{R})$ is $\{z_1, z_2\}$

where $z_1 = 1/\sqrt{2}$ $\qquad z_2 = \sqrt{3/2}\, x$

Now
$$[T]_{\tilde{\beta}}^{\tilde{\gamma}} = \begin{pmatrix} 0 & 0 & \dfrac{2\sqrt{45}}{\sqrt{19 - 10\sqrt{2}}} \\ 0 & 0 & 0 \end{pmatrix}$$

and so $[T^*]_{\tilde{\gamma}}^{\tilde{\beta}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{2\sqrt{45}}{\sqrt{19-10\sqrt{2}}} & 0 \end{pmatrix}$

Thus $[T^*T]_{\tilde{\beta}}^{\tilde{\beta}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{2\sqrt{45}}{\sqrt{19-10\sqrt{2}}} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{2\sqrt{45}}{\sqrt{19-10\sqrt{2}}} \\ 0 & 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{180}{19-10\sqrt{2}} \end{pmatrix}$$

To construct the Singular Value Decomposition of T, we should pick an o.n. basis of eigenvectors for T^*T.
So $\tilde{\beta}$ will do! (the matrix of T^*T w.r.t. $\tilde{\beta}$ is already diagonal, so $\tilde{\beta}$ is a basis of eigenvectors, and $\tilde{\beta}$ is orthonormal by construction).
Actually, we should re-order so that the non-zero eigenvalue corresponds to the <u>first</u> eigenvector:

$$\beta = \{ y_3, y_1, y_2 \}$$

Now $T(y_3) = \frac{2\sqrt{45}}{\sqrt{38-20\sqrt{2}}} \cdot 1$

and we set $g_1 = \frac{T(y_3)}{\|T(y_3)\|} = 1/\sqrt{2}$.

Extending $\{g_1\}$ to an o.n. basis $\gamma = \{\frac{1}{\sqrt{2}}, \sqrt{3/2}\, x\}$ for $P_1(\mathbb{R})$ gives

$$[T]_\beta^\gamma = \begin{pmatrix} \frac{2\sqrt{45}}{\sqrt{19-10\sqrt{2}}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of T is 1, and the non-zero singular value $\sigma_1 = \dfrac{2\sqrt{45}}{\sqrt{19-10\sqrt{2}}}$

[Sorry about the horrible numbers: I made a calculational error when setting the question.]

(d) The only one-dimensional T-invariant subspace of $\mathbb{R}^3$ is span $\{(1,1,1)\}$, so any eigenvector of T must be a scalar multiple of $(1,1,1)$. If T were self-adjoint, there would be an o.n. basis for $\mathbb{R}^3$ consisting of eigenvectors for T, so T is not self-adjoint. However, T is length- and angle-preserving, so T is orthogonal.

#2 See class notes / textbook.

#3 (a) We will find a basis $\beta = \{p_1(x), p_2(x)\}$ for $P_1(\mathbb{R})$ such that $\{f_1, f_2\}$ is the basis dual to β. This proves (by Thm 2.24) that $\{f_1, f_2\}$ is a basis for V^*.

Let
$$p_1(x) = a + bx$$
$$p_2(x) = c + dx$$

We need:
$$f_1(p_1) = 1 \Rightarrow a + b/2 = 1$$
$$f_1(p_2) = 0 \Rightarrow 2a + 2b = 0$$
$$f_2(p_1) = 0 \Rightarrow c + d/2 = 0$$
$$f_2(p_2) = 1 \Rightarrow 2c + 2d = 1$$

So $a = 2,\ b = -2,\ c = -1/2,\ d = 1$

and $\beta = \{2 - 2x,\ -\frac{1}{2} + x\}$.

(b) Case I: V is finite-dimensional.

If $f = 0_{V^*}$ then $g = 0_{V^*}$ too and we can take α = anything.
Otherwise, $R(f) = \mathbb{C}$ so $\text{rank}(f) = 1$
$\Rightarrow \text{nullity}(f) = n - 1$ where $n = \dim V$ (Dimension Theorem)

Pick a basis $\{v_1, \dots, v_{n-1}\}$ for $N(f)$. Since $N(f) = N(g)$, this is also a basis for $N(g)$. Extend to a basis $\beta = \{v_1, \dots, v_{n-1}, v\}$ for V.
Let $\beta^* = \{f_1, \dots, f_{n-1}, h\}$ be the basis for V^* dual to β.

Then $\quad f = a_1 f_1 + \cdots + a_{n-1} f_{n-1} + bh$

$\Rightarrow \quad a_i = f(v_i) = 0 \quad$ as $\quad v_i \in N(f)$

$\Rightarrow \quad f = bh \qquad [b \neq 0$ as $f \neq 0_{V^*}]$

$b' \neq 0$ either

Similarly $\quad g = b'h$, so $f = \alpha g$ where $\alpha = b/b'$.

Case II : V is infinite-dimensional.

If $f = 0_{V^*}$ then we are done (same argument again).

Otherwise $\exists\ x \in V$ with $f(x) \neq 0$ and hence $g(x) \neq 0$.

~~Then~~ Claim : $\quad g(y) = \dfrac{g(x)}{f(x)} f(y) \quad \forall\ y \in V$

Proof : Suppose not. Then $\exists\ y' \in V$ with

$$g(y') \neq \frac{g(x)}{f(x)} f(y')$$

Consider $W = \text{span}\{x, y'\}$ and restrict f and g to give maps $f|_W \in W^*$, $g|_W \in W^*$.

$f|_W(w) = 0 \iff g|_W(w) = 0$, so we can apply Case I to conclude that

$g|_W(w) = \alpha f|_W(w) \quad \forall\ w \in W$

for some $\alpha \in \mathbb{C}$. Putting $w = x$ we see that $\alpha = \dfrac{g(x)}{f(x)}$.

But then $g|_W(y') \neq \alpha f|_W(y')$. $\#$

This proves the Claim.

Taking $\alpha = g(x)/f(x)$ proves Case II. We are done. $\square$

#4 : (a) The auxiliary polynomial is $x^2 - 2ax + a^2 + b^2$

which has roots $x = \dfrac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2}$

$= a \pm ib$

assume $b \neq 0$ throughout

By the Corollary to ~~the~~ Th$^{\underline{m}}$ 2.33, we know that a basis for the space of solutions in $C^\infty(\mathbb{C})$ is

$$\{ \exp((a+ib)t), \ \exp((a-ib)t) \}$$

We seek <u>real</u> solutions (ie solutions $y \in C^\infty(\mathbb{C})$ such that $y(t) \in \mathbb{R}$ for all $t \in \mathbb{R}$). We know that this will give us all the solutions in $C^\infty(\mathbb{R})$ because $C^\infty(\mathbb{R})$ is a subspace of $C^\infty(\mathbb{C})$ (where we regard $C^\infty(\mathbb{C})$ as a vector space over $\mathbb{R}$ in the natural way). So we seek

$$(\star) \qquad y = (A+iB)\exp((a+ib)t) + ((C+iD)\exp((a-ib)t)$$
$$= e^{at}(A+iB)(\cos(bt) + i\sin(bt)) + e^{a}(C+iD)(\cos(bt) - i\sin(bt))$$

such that $y(t)$ is real for all t. The imaginary part of $(\star)$ is

$$e^{at}\left(A \overset{\sin}{\cancel{\cos}}(bt) + B\cos(bt) \right\} + D\cos(bt) - C\sin bt \Big)$$

For this to be real for all $t \in \mathbb{R}$, we need

$$(A - C)\sin bt + (B + D)\cos(bt) = 0 \quad \text{for all } t$$

$\Rightarrow A = C, \ B = -D$ as the set $\{\cos bt, \sin bt\}$ is LI in $C^\infty(\mathbb{R})$ [if $b \neq 0$]

So

$$y = e^{at}\left(A\cos(bt) - B\sin(bt) + A\cos bt - B\sin bt\right)$$

$$= 2Ae^{at}\cos(bt) \mp 2B\, e^{at}\sin(bt)$$

In other words $y \in \text{span}\{e^{at}\cos bt,\ e^{at}\sin bt\}$

Since both $e^{at}\cos bt$ and $e^{at}\sin bt$ are in the solution space [take $A = C = \frac{1}{2}$, $B = D = 0$ or $A = C = 0$, $B = -\frac{1}{2}$, $D = \frac{1}{2}$] we see that the solution space in $C^\infty(\mathbb{R})$ is $\text{span}\{e^{at}\cos(bt),\ e^{at}\sin(bt)\}$.

(b) We know $y = Ae^{-t}\cos 3t + De^{-t}\sin(3t)$, and

that $y(0) = 0 \Rightarrow A = 0$

and $y'(0) = v \Rightarrow 3B = v$

so $y(t) = \frac{v}{3}e^{-t}\sin(3t)$.

As $t \to \infty$, $y(t) \to 0$ i.e. the amplitude of the oscillation decreases to zero with time.

#5 : Claim : Let δ be a basis for V such that $[T]_\delta^\delta$ is in Jordan canonical form

with $n_1(\lambda)$ blocks of size 1 with gen. e'value λ

$n_2(\lambda)$ blocks of size 2 with gen. e'value λ

$\vdots$

$n_k(\lambda)$ blocks of size k with gen. e'value λ

$\vdots$

Then
$$\text{nullity}(T-\lambda I) = n_1 + \cdots + n_k + \cdots$$
$$\text{nullity}(T-\lambda I)^2 = n_1 + 2n_2 + 2n_3 + \cdots + 2n_k + \cdots$$
$$\text{nullity}(T-\lambda I)^3 = n_1 + 2n_2 + 3n_3 + \cdots + 3n_k + \cdots$$
$$\vdots$$

Proof :
$$\text{nullity}(T-\lambda I) = \text{nullity}([T]_\delta^\delta - \lambda I)$$
$$= \text{nullity}\begin{pmatrix} A_1-\lambda I & O & O \\ O & \ddots & O \\ O & O & A_r-\lambda I \end{pmatrix}$$

where $A_1, \ldots, A_r$ are the Jordan blocks

$$= \text{nullity}(A_1-\lambda I) + \cdots + \text{nullity}(A_r-\lambda I)$$

But
$$\text{nullity}(A_j-\lambda I) = \text{nullity}\begin{pmatrix} \lambda_j-\lambda & 1 & & O \\ & \lambda_j-\lambda & 1 & \\ & & \ddots & 1 \\ O & & & \lambda_j-\lambda \end{pmatrix} = \begin{cases} 1 & \text{if } \lambda_j = \\ 0 & \text{else} \end{cases}$$

so nullity $(T-\lambda I)$ = # of blocks with generalized eigenvalue λ

$$= \underbrace{n_1(\lambda) + n_2(\lambda) + \cdots + n_k(\lambda) + \cdots}$$

this looks like an infinite sum but in fact isn't, as there are no blocks of size bigger than $\mathrm{mult}(\lambda)$, so $n_{k'}(\lambda) = 0$ for all $k' > \mathrm{mult}(\lambda)$.

Similarly, nullity $(T-\lambda I)^2$ = nullity $(A_1 - \lambda I)^2 + \cdots$ + nullity $(A_r - \lambda I)^2$

and nullity $(A_j - \lambda I)^2 = 0$ if $\lambda_j \neq \lambda$

[the diagonal entries of $(A_j - \lambda I)^2$ will be non-zero]

and it is upper-triangular

and nullity $(A_j - \lambda I)^2$ = nullity $\begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \vdots \\ & & & & \ddots & 0 \\ & & & & & 1 \\ & & & & & \end{pmatrix}$

$= 2$ if the block has size at least 2

and nullity $(A_j - \lambda I)^2$ = ~~nullity~~ $\cancel{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2}$

nullity $(0)^2 = 1$ if the block has size 1

so nullity $(T-\lambda I)^2 = 1 \cdot$ (# of blocks of size 1 for λ)

$+ 2 \cdot$ (# of blocks of size ≥ 2 for λ)

$$= n_1(\lambda) + 2n_2(\lambda) + 2n_3(\lambda) + \cdots + 2n_k(\lambda) + \cdots$$

Repeating for $(T-\lambda I)^3, (T-\lambda I)^4, \ldots$ etc. proves the Claim. $\square$

Thus $n_1(\lambda) = 2\,\text{nullity}(T-\lambda I) - \text{nullity}(T-\lambda I)^2$

$$n_2(\lambda) = \frac{3}{2}\left(\text{nullity}(T-\lambda I)^2 - n_1(\lambda)\right) - \left(\text{nullity}(T-\lambda I)^3 - n_1(\lambda)\right)$$

$$\vdots$$

and so the numbers $n_1(\lambda), n_2(\lambda), \ldots$ can be uniquely determined from the numbers

$$\text{nullity}(T-\lambda I), \quad \text{nullity}(T-\lambda I)^2, \ldots$$

But these do not depend on the choice of Jordan canonical basis δ, in other words

$$[T]_\beta^\beta \quad \text{and} \quad [T]_\gamma^\gamma$$

have the same # of Jordan blocks of each size for each generalized eigenvalue λ. They therefore differ only by a reordering of their Jordan blocks.

(b) Let $\gamma = \{1, x, x^2, x^3\}$ be the standard basis for $P_3(\mathbb{R})$. Then

$$[T]_\gamma^\gamma = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

The only eigenvalue of T is 2 and

$$N(T-2I) = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$N\left((T-2I)^2\right) = N\left(\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 \right) = N \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= P_3(\mathbb{R})$$

Pick $v \in N((T-2I)^2)$, $v \notin N(T-2I)$: say $v = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Then $\{(T-2I)v, v\} = \left\{ \begin{pmatrix} 0 \\ 6 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a cycle of gen. eigenvectors.

Now try to pick $w \in N(T-2I)^2$, $\cancel{w \notin \operatorname{span}\{(T-2I)v, v\}}$

$w \notin \operatorname{span}\left(\{(T-2I)v, v\} \cup N(T-2I)\right) = \operatorname{span} \left\{ \begin{pmatrix} 0 \\ 6 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$.

Take, for example, $w = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. Then $\{(T-2I)w, w\} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

is another cycle of gen. eigenvectors and

$$\beta = \left\{ \begin{pmatrix} 0 \\ 6 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$ ← co-ordinate vectors [w.r.t. γ] for these polynomials

$$= \{ 6x, x^3, 2, x^2 \}$$

is a basis for $P_3(\mathbb{R})$ with respect to which

$$[T]_\beta^\beta = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(c) $$\delta = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/6 \end{pmatrix}, \begin{pmatrix} 200 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 100 \\ 0 \end{pmatrix} \right\}$$

$$\updownarrow$$

$$\left\{ x, \frac{x^3}{6}, 200, 100x^2 \right\}$$

will do.

6

(a) We need to show $(R+\lambda S)^*(x) = R^*(x) + \bar{\lambda} S^*(x)$

for all $x \in V$, so it is enough to show that

$$\langle (R+\lambda S)^*(x), y \rangle = \langle R^*(x) + \bar{\lambda} S^*(x), y \rangle$$

for all $x, y \in V$

But $\langle (R+\lambda S)^*(x), y \rangle = \langle x, R(y) + \lambda S(y) \rangle$

$$= \langle x, R(y) \rangle + \bar{\lambda} \langle x, S(y) \rangle$$

$$= \langle R^*(x), y \rangle + \bar{\lambda} \langle S^*(x), y \rangle$$

$$= \langle R^*(x) + \bar{\lambda} S^*(x), y \rangle,$$

for any $x, y \in V$.

(b) $\|T(x)\|^2 = \langle Tx, Tx \rangle = \langle x, T^*Tx \rangle$

$$= \langle x, TT^*x \rangle$$

$$= \langle T^*x, T^*x \rangle$$

$$= \|T^*x\|^2$$

Since $\|Tx\|$ and $\|T^*x\|$ are both non-negative, we see that $\|Tx\| = \|T^*x\|$.

(c) It is enough to show (see the hint) that $\|(T-\lambda I)(x)\| = 0 \iff \|(T^* - \bar{\lambda} I)(x)\| = 0$

But by (a) and (b), $\|(T-\lambda I)(x)\| = \|(T^* - \bar{\lambda} I)(x)\|$.

#6(d) We will prove this by induction on $n = \dim V$.

Base case : $n = 1$

This is trivial.

Induction step : Assume that for every $(n-1)$-dim'l i.p. space $\tilde{V}$ and every normal linear transformation $\tilde{T} : \tilde{V} \to \tilde{V}$, there is an o.n. basis for $\tilde{V}$ consisting of eigenvectors for $\tilde{T}$.

Since we are working over $\mathbb{C}$, the char. poly. of T splits and so $\exists$ an eigenvector $v \in V$ for T. Let $W = \text{span}\{v\}$.

W is T-invariant $\Rightarrow$ $W^\perp$ is T^*-invariant (from class)

W is also T^*-invariant, since v is an eigenvector for T^* (by (c)), so $W^\perp$ is T^{**}-invariant
$\Rightarrow$ $W^\perp$ is T-invariant.

Let $\tilde{V} = W^\perp$ and $\tilde{T} = T|_{W^\perp}$.

$\tilde{V}$ is an $(n-1)$-dim'l inner product space, and

$\tilde{T}^* = (T|_{W^\perp})^* = T^*|_{W^\perp}$ so $\tilde{T}$ is

normal :
$$\tilde{T}\tilde{T}^* = T|_{W^\perp} T^*|_{W^\perp}$$
$$= (TT^*)|_{W^\perp}$$
$$= (T^*T)|_{W^\perp}$$
$$= T^*|_{W^\perp} T|_{W^\perp}$$
$$= \tilde{T}^*\tilde{T}$$

By the induction hypothesis, $\exists$ an o.n. basis for $\tilde{V}$ consisting of eigenvectors for $\tilde{T}$, call it $\{v_1, \dots, v_{n-1}\}$. Then $\{v_1, \dots, v_{n-1}, v\}$ is an o.n. basis for V consisting of e'vectors for T.

By induction, we are done $\square$

(e) $V = \mathbb{C}^2$, $T(x,y) = (ix, 3iy)$.

This has ~~eigenvectors~~ non-real eigenvalues, so T cannot be self adjoint, but $T^*(x,y) = (-ix, -3iy)$

so $T^*T(x,y) = \cancel{(x,y)}\ (x, 9y) = TT^*(x,y)$

#7 (a) If T is nilpotent, say $T^k = 0$, and $Tx = \lambda x$ then

$$0 = T^k(x) = \lambda^k \{x\}$$

$\Rightarrow \lambda = 0$. So every eigenvalue of a nilpotent linear operator is zero.

Conversely, if every eigenvalue of T is zero then for a Jordan canonical basis β $[T]_\beta^\beta$ is upper-triangular with zeroes on the diagonal. Thus $([T]_\beta^\beta)^n = 0$, where $n = \dim V$, so $T^n = 0$.

(b) If $A = [T]_\beta^\beta$ and $B = [T]_\gamma^\gamma$ where β and γ are different ~~matrices~~ bases for V then $B = Q^{-1}AQ$ where $Q = [Id]_\gamma^\beta$

Thus, by Q1(a), $tr(B) = tr(A)$.

(c) Choose a basis β for V such that

$[T]_\beta^\beta$ is upper-triangular

(for example β could be a Jordan

canonical basis). Then

$$[T]_\beta^\beta = \begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ & \lambda_2 & * & \cdots & * \\ & & \ddots & & \vdots \\ & 0 & & & * \\ & & & & \lambda_n \end{pmatrix}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues for T (not necessarily distinct). So $\mathrm{tr}(T) = \lambda_1 + \cdots + \lambda_n$.

Also $\left([T]_\beta^\beta\right)^2 = \begin{pmatrix} \lambda_1^2 & * & \cdots & * \\ 0 & \lambda_2^2 & \cdots & * \\ 0 & 0 & \ddots & * \\ \vdots & \vdots & & * \\ 0 & 0 & \cdots\ 0 & \lambda_n^2 \end{pmatrix}$

so $\mathrm{tr}(T^2) = \lambda_1^2 + \cdots + \lambda_n^2$

$\vdots$

$\mathrm{tr}(T^n) = \lambda_1^n + \cdots + \lambda_n^n$

Thus if T is nilpotent, so $\lambda_1 = \cdots = \lambda_n = 0$, then $\mathrm{tr}(T) = \mathrm{tr}(T^2) = \cdots = \mathrm{tr}(T^n) = 0$.

Conversely, if $\mathrm{tr}(T) = \cdots = \mathrm{tr}(T^n) = 0$ then we know that

$$\begin{aligned} \lambda_1 + \cdots + \lambda_n &= 0 \\ \lambda_1^2 + \cdots + \lambda_n^2 &= 0 \\ &\vdots \\ \lambda_1^n + \cdots + \lambda_n^n &= 0 \end{aligned}$$

and we need to show that $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$

[as then T will be nilpotent by (a)].

This is true, but it is <u>much</u> harder to prove then I had thought (and certainly ~~[illegible]~~ far too hard to appear on the final).

Firstly, note that

$$\lambda_1 = \cdots = \lambda_n = 0 \iff (x-\lambda_1)\cdots(x-\lambda_n) = x^n$$

$$\iff x^n - (\lambda_1 + \cdots + \lambda_n)x^{n-1} + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots)x^{n-2} + \cdots + (\lambda_1\lambda_2\cdots\lambda_n) = x^n$$

$$\iff \begin{cases} \lambda_1 + \cdots + \lambda_n = 0 \\ \sum_{i<j} \lambda_i\lambda_j = 0 \\ \sum_{i<j<k} \lambda_i\lambda_j\lambda_k = 0 \\ \vdots \\ \lambda_1\lambda_2\cdots\lambda_n = 0 \end{cases}$$

these are called THE ELEMENTARY SYMMETRIC FUNCTIONS of $\lambda_1, \ldots, \lambda_n$.

So it suffices to show that

$$\lambda_1 + \cdots + \lambda_n = 0$$

$$\sum_{i<j} \lambda_i\lambda_j = 0$$

$$\vdots$$

$$\lambda_1\lambda_2\cdots\lambda_n = 0$$

Consider the generating function

$$p(t) = (\lambda_1 + \dots + \lambda_n) + (\lambda_1^2 + \dots + \lambda_n^2)t + \dots$$

$$= \sum_{i=1}^{i=n} \sum_{r \geq 1} \lambda_i^{r} t^{r-1}$$

$$= \sum_{i=1}^{i=n} \frac{\lambda_i}{1-\lambda_i t}$$

Then if $h(t) = \prod_{i=1}^{i=n} \frac{1}{1-\lambda_i t}$

then $\quad d/dt\,(\log h(t)) = -\sum_{i=1}^{i=n} \frac{\lambda_i}{1-\lambda_i t}$

and so $\quad p(t) = -d/dt \log h(t)$

If $\lambda_1 + \dots + \lambda_n = 0$ and $\lambda_1^2 + \dots + \lambda_n^2 = 0$ and ...

then $p(t) = 0 \quad \Rightarrow \quad \log h(t) = \text{constant}$ (independent of t)

$\Rightarrow \quad h(t)$ is independent of t

But $h(0) = 1$ [regardless of the value of the λ_i] and

so $h(t) = 1$.

Thus $\frac{1}{h(t)} = 1$ too, so $(1-\lambda_1 t)\dots(1-\lambda_n t) = 1$

$$\Rightarrow 1 - (\lambda_1 + \dots + \lambda_n)t + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \dots)t^2 - \dots = 1$$

$$\Rightarrow \lambda_1 + \dots + \lambda_n = 0$$
$$\sum_{i<j} \lambda_i \lambda_j = 0$$
etc.

$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0$

It remains to show that <u>if</u> $\lambda_1 + \dots + \lambda_n = 0$

and $\lambda_1^2 + \dots + \lambda_n^2 = 0$

and $\dots$

and $\lambda_1^n + \dots + \lambda_n^n = 0$

<u>then</u> $\lambda_1^k + \lambda_2^k + \dots + \lambda_n^k = 0$ for $k = n+1, n+2, \dots$

But this is $\mathrm{tr}(T^k)$.

Cayley-Hamilton implies that

$$T^n = \alpha_0 I + \alpha_1 T + \dots + \alpha_{n-1} T^{n-1}$$ for some $\alpha_0, \dots, \alpha_{n-1} \in \mathbb{C}$

$$\Rightarrow \quad T^{n+1} = \alpha_0 T + \alpha_1 T^2 + \dots + \alpha_{n-1} T^n$$

$$\Rightarrow \quad \mathrm{tr}(T^{n+1}) = \alpha_0 \mathrm{tr}(T) + \dots + \alpha_{n-1} \mathrm{tr}(T^n)$$
$$= 0$$

Similarly $$T^{n+2} = \alpha_0 T^2 + \alpha_1 T^3 + \dots + \alpha_{n-1} T^{n+1}$$
$$= \tilde{\alpha}_0 T + \tilde{\alpha}_1 T^2 + \dots + \tilde{\alpha}_{n-1} T^n$$

$$\Rightarrow \quad \mathrm{tr}(T^{n+2}) = 0$$ too

etc.

So if $\mathrm{tr}(T) = \mathrm{tr}(T^2) = \dots = \mathrm{tr}(T^n) = 0$

then all eigenvalues of T are zero, and hence T is nilpotent.