1 Section 6.3

1.1 Problem 3

To find T^*, we merely need to write down the matrix of T (in the first two cases with respect to the standard basis) and find its conjugate transpose.

1.1.1 (a)

$T^*(x) = (11, -12)$

1.1.2 (b)

$T^*(x) = (5 + i, -1 - 3i)$

1.1.3 (c)

It is easy to check using the given inner product that $\{\sqrt{2}/2, t\sqrt{6}/2\}$ is an orthonormal basis for $P_1(\mathbb{R})$ and the matrix for T with respect to this basis is:

$$\begin{bmatrix} 3 & \sqrt{3} \\ 0 & 3 \end{bmatrix}$$

and so $T^*(f(t)) = 12 + 6t$.

1.2 Problem 6

For all x, y,

$< U_1x, y >=< (T + T^*)x, y >=< Tx, y > + < T^*x, y >=< x, T^*y > + < x, Ty >=< x, (T + T^*)y >=< x, U_1y >\Rightarrow U_1 = U_1^*$.

$< U_2x, y >=< T((T^*)x(, y >=< T^*x, T^*y >=< x, T(T^*y) >=< x, U_2y >\Rightarrow U_2 = U_2^*$.

1.3 Problem 11

For all $v \in V$

$0 =< T^*Tv, v >=< Tv, Tv >\Rightarrow Tv = 0 \forall v \in V \Rightarrow T = T_0$.

The second part follows from the obvious equality $T^*T = TT^*$.

1.4 Problem 15

1.4.1 (a)

Assume S and R are adjoints for T. Then $< x, S(y) >_1=< x, R(y) >_1> \forall x \in V, y \in W$ and so by taking inner products with the standard basis, we have $R(y) = S(y) \forall y \in W$ which shows $R = S$.

Let $S = T^*, x \in V, y, z \in W$ and note $< x, S(cy+z) >=< Tx, cy+z >= c < x, Sy > + < x, Sz >$ linearity and the adjoint relationship. Since this is true for all $x \in V$ by taking the inner product with an orthonormal basis we have $S(cy+z) = cS(y) + S(z) \Rightarrow S$ is linear.

1.4.2 (b)

Let $\beta = \{b_i\}$ be a basis for V, $\gamma = \{g_i\}$ a basis for W, $A = [T]_\beta^\gamma$, and $B = [T^*]_\gamma^\beta$.

$T(b_j) = \sum_1^m < T(b_j), g_i > g_i$ so $A_{ij} =< T(b_j), g_i >$.

Likewise $B_{ij} =< T^*(g_j), b_i >=< g_j, T(b_i) >= \overline{(A_{ji})}$. So $B = A^*$.

1.4.3 (c)

Using the notation of the previous problem, it suffices to show rank B = rank A. Since $B = A^*$, by the equivalence of column rank and row rank it suffices to show that for an arbitrary set of vectors $\{x_i\}$ that dim (span $\{x_i\}$)= dim (span $\{\bar{x_i}\}$). Write the last k vectors as linear combinations of the first n, and then it is easy to check that the first n of the $\{\bar{x_i}\}$ are linearly independent and span the last k.

2 Section 6.7

2.1 Problem 3

2.1.1 (a)

$$A * A = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

It is easy to check that the eigenvalues of A^*A are $\lambda_1 = 6, \lambda_2 = 0$ with orthonormal eigenvectors $v_1 = (\sqrt{2}/2, \sqrt{2}/2)^t$ and $v_2 = (\sqrt{2}/2, -\sqrt{2}/2)^t$. So the only nonzero singular value is $\sigma_1 = \sqrt{6}$ and we have that $V = (v_1, v_2)$ and

$$\Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

As in Theorem 6.27, take $u_1 = (1/\sigma_1)L_A(v_1) = (\sqrt{3}/3)(1,1,-1)^t$ and choose vectors $u_2 = (\sqrt{2}/2)(1,-1,0)^t$ and $u_3 = (\sqrt{6}/6)(1,1,2)^t$

so that $\{u_i\}$ is an orthonormal basis. Let $U = (u_1, u_2, u_3)$ and then by matrix multiplication, $A = U\Sigma V$.

2.1.2 (b)

Similar to (a)

2.2 Problem 9

Define $\sigma_n = 0 \forall n > r$

2.2.1 (a)

$< T^*(u_i), v_j >=< u_i, T(v_j) >=< u_i, \sigma_j u_j >= \delta_{ij}$ so $T^*(u_i) = \sum_{j=1}^n < T^*(u_i), v_j > v_j = \delta_{ij} v_i \Rightarrow TT^*(u_i) = \sigma_i^2 u_i$ as desired.

2.2.2 (b)

This follows easily from combining Theorem 6.27 and part (a) to show that the $sigma_i$ of (a) are the singular values of A.

2.2.3 (c)

From 6.26 and (a), these eigenvalues are both equal to the σ_i^2.

2.2.4 (d)

A^*A and AA^* have the same eigenvalues. This follows because the eigenvalues of a matrix are equal to those of the linear transformation.

2.3 Problem 17

For this problem ONLY, represent the dagger symbol by $*$ w.r.t the standard basis, let $A = [U], B = [T]$. Then $AB = [UT]$ and so we can solve (a) and (b) simultaneously by showing $(AB)^* \neq B^*A^*$. Finding these matrices we have:

$$(AB)^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} = B^*A^*$$

2.4 Problem 21

2.4.1 (a)

Take $x \in N(T)^\perp$ with $T(x) = y$. Then $TT^\dagger T(x) = TT^\dagger(y) = T(x)$. But if $x \in N(T)$ then $TT^\dagger T(x) = TT^\dagger(0) = 0 = T(x)$. So $TT^\dagger T = T$.

2.4.2 (b)

Take $y \in R(T)$ with $T(x) = y$. Then $T^\dagger TT^\dagger(y) = T^\dagger T(x) = T^\dagger(y)$ and if $y \in R(T)^\perp$ then $T^\dagger TT^\dagger(y) = T^\dagger T(0) = 0 = T^\dagger(y)$

2.4.3 (c)

Let β and γ be the orthonormal bases for V and W respectively that come from the splittings $V = N(T)^{\perp} \oplus N(T)$ and $W = R(T)^{\perp} \oplus R(T)$. Then $[T^{\dagger}T]_{\beta}$ and $[TT^{\dagger}]_{\gamma}$ each have the form $I_r \oplus 0_k$ for some r, k and are diagonal; therefore $T^{\dagger}T$ and $TT^{\dagger}$ are self-adjoint.

2.5 Problem 22

It is sufficient to show that $UT(x) = x \forall x \in N(T)^{\perp}$ and $U(y) = 0 \forall y \in R(T)^{\perp}$

(1): let $z \in N(T)$ and then $< UT(x), z >=< x, (UT)^*(z) >=< x, UT(z) >=< x, 0 >= 0$ shows $UT(x) \in N(T)^{\perp} \forall x \in V$. Then $TUT(x) = T(x) \Rightarrow UT(x) = x$ because if $y \in N(T)^{\perp}$ and $T(y) = T(x)$ it is easy to check that $y = x$.

(2): let $y \in R(T)^{\perp}$ and then $< TU(y), T(x) >=< y, (TU)^*(T(x)) >=< y, TUT(x) >=< y, T(x) >= 0 \Rightarrow TU(y) = 0$ because it is in $R(T)$ so then we have $U(y) \in N(T)$. But if $x \in N(T)$ then $< U(y), UT(x) >=< U(y), U(0) >= 0$ and also equals $< (UT)^* U(y), x >=< U(y), x > \Rightarrow U(y) \in N(T)^{\perp} \Rightarrow U(y) = 0$.