1 Section 5.2

1.1 Problem 2

1.1.1 (d)

Let A be the given matrix. det $(A - tI) = -(t-3)^2(t+1)$ (which splits) and so we have eigenvalues $\lambda_1 = 3$ and $\lambda_3 = -1$. By solving the equations for the nullspace of $(A - \lambda_i I)$ we find that there are eigenvectors $v_1 = (1,1,0)^t$, $v_1' = (0,0,1)^t$ and $v_2 = (1,2,3/2)^t$, with the first two being LI eigenvectors of $\lambda_1 = 3$. So the algebraic and geometric multiplicities are equal for each eigenvalue and so $A = Q^{-1}DQ$ where D has diagonal entries 3, 3, and 1 and $Q = (v_1, v_1', v_2)$ is the matrix of eigenvectors.

1.1.2 (f)

The char. poly. in this case is $(1-t)^2(3-t)$ so 1 is an eigenvalue with algebraic multiplicity 2. However if we solve for $N(A-I)$ we find that it is spanned by $(1,0,0)^t$ and therefore 1 has geometric multiplicity 1. So A is not diagonalizable.

1.2 Problem 7

We want to diagonalize A as $A = Q^{-1}DQ$, where Q is a matrix whose columns are the eigenvectors of A, so that $A^n = Q^{-1}D^nQ$. det $(A - tI) = t^2 - 4t - 5 = (t-5)(t+1)$ and by computing the nullspace of $A - \lambda I$ for $\lambda = 5$ and $\lambda = 1$ (this is a simple exercise in a system of two equations in two unknowns) we find

$$Q = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

and so

$$Q^{-1} = \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

and

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

$\Rightarrow$

$$D^n = \begin{bmatrix} (-1)^n & 0 \\ 0 & 5^n \end{bmatrix}$$

so A^n is given as above.

2 Section 5.4

2.1 Problem 13

Let $w \in W$. Then $w = a_1 v + \cdots + a_n T^n(v)$. So $w = g(T)v$ where $g(t) = \sum_{i=1}^{i=n} a_i t^i$.

Let $w = g(T)v$ and $g(t)$ be as above. Then $w = \sum_{i=1}^{i=n} a_i T^i(v) \Rightarrow w \in W$.

2.2 Problem 15

Let $A = [T]_\beta$ where $T : V \to V$ and β is a basis for V. Then since the characteristic polynomial is invariant, it is $f(t) = \det(A - tI)$. Then $f(A) = f([T]_\beta) = [f(T)]_\beta$ by linearity. By Cayley-Hamilton for transformations $f(T) = T_0$ and so $f(A) = [0]_\beta = 0$.

2.3 Problem 17

Assume WLOG that the first $n+1$ terms in this sequence are LI, so $a_0 I_n + a_1 A + \cdots + a_n A^n = 0 \Rightarrow a_i = 0\ \forall i$. Let $v \in \mathbb{F}^n$ and note that $0(v) = 0 \Rightarrow a_0(v) + a_1(Av) + \cdots + a_n(A^n v) = 0$. If there were $\{a_i\}$ not all equal to 0 that solved this last equation this would contradict the above statement that $a_i = 0\ \forall i$; however this implies that there are $n+1$ LI vectors $\{v, Av, \ldots, A^n v\}$ in $\mathbb{F}^n$, a contradiction. Therefore the dimension of this span must be $\leq n$.

2.4 Problem 21

Let V not be a T-cyclic subspace of itself. Then for any $v \in V$, $Tv = cv$ (because otherwise v, Tv is a basis for V); furthermore note that all such c are the same (because if $Tv_1 = c_1 v_1$ and $Tv_2 = c_2 v_2$ then for $v = v_1 + v_2$ we again have v, Tv is a basis for V). So $T = cI$ unless V is a T-cyclic subspace of itself.

2.5 Problem 27

2.5.1 (a)

If $v + W = v' + W$ then $v - v' \in W \Rightarrow T(v) - T(v') \in W \Rightarrow \bar{T}(v + W) = \bar{T}(v' + W)$, i.e. $\bar{T}$ is well-defined.

2.5.2 (b)

$\bar{T}$ clearly inherits linearity from T

2.5.3 (c)

$\bar{T}\eta(v) = \bar{T}(v + W) = T(v) + W = \eta(T(v))$

2.6 Problem 28

Let γ and β be as given in the hint. By previous homework, (1.6 Problem 35) the given set α is a basis for V/W. Now since W is T-invariant, we know

$$[T]_\beta = \begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix}$$

where $B_1 = [T|_W]_\gamma$. Let $i > k$ and furthermore note $\bar{T}(v_i + W) = T(v_i) + W$ and in the basis β, the first k terms of $T(v_i)$ are in W and therefore $\bar{T}(v_i + W) = \sum_{j=k+1}^{n} b_j v_j + W$

where b_j is the (j,i)th entry of $[T]_\beta$. This shows $[\bar{T}]_\alpha = B_3$. So by computing the determinant (say byexpanding along the first column) we have that det $([T]_\beta - tI)$ det $([T|_W]_\gamma - tI)$ det $([\bar{T}]_\alpha - tI)$ i.e. $f(t) = g(t)h(t)$.

2.7 Problem 29

T is diagonalizable, so its char. poly. splits, and the algebraic and geometric multiplicity of its eigenvalues are equal. From 28, clearly if the char. poly. of T splits than that of $\bar{T}$ does also. Furthermore if λ is an eigenvalue of $\bar{T}$, it is also an eigenvalue of T and so the algebraic and geometric multiplicities are inherited from T and are again equal. So $\bar{T}$ is diagonalizable.

2.8 Problem 30

Using the notation of Problem 28, clearly g and h split, so $f = gh$ splits and it remains to show that the algebraic and geometric multiplicities of the eigenvalues of T are equal. Let λ be an eigenvalue. Then clearly its algebraic multiplicity as an eigenvalue of T is equal to that multiplicity as an eigenvalue of either T_W or $\bar{T}$ since $f = gh$. Furthermore its geometric multiplicity as an eigenvalue of T_W or $\bar{T}$ must be at least as large as its geometric multiplicity as an eigenvalue of T, since the generalized eigenspace of the eigenvalue w.r.t. T_W or $\bar{T}$ is a subspace of the generalized eigenspace of the eigenvalue w.r.t. T. So T is diagonalizable.