$\boxed{5.2.1}$

(a) False. Eg: identity operator

(b) False. See 5.1.1

(c) False. Eg: the zero vector.

(d) True. (Thm 5.5)

(e) False. (Corollary to Thm. 2.3)

(f) False. polynomial splitting also essential

(g) True.

(h) True.

(i) False

$\boxed{5.2.8}$ (Thanks to Kelly Shue - an adapted proof)

$A \in M_{n\times n}(F)$ has 2 distinct eigenvalues λ_1, λ_2, $\dim(E_{\lambda_1}) = n-1$.

Then A diagonalizable

$A \leftrightarrow$ linear operator L_A.

★ Careful! You cannot assume λ_1, λ_2 are only eigenvalues, or that T splits over $\mathbb{F}$.

Need: (i) $f(t)$ splits over F

(ii) $\dim(E_{\lambda_1}) = \text{mult}(\lambda_1) \overset{\text{not.}}{=} m_1$

$\dim(E_{\lambda_2}) = \text{mult}(\lambda_2) \overset{\text{not.}}{=} m_2$.

By Thm. 5.7, $\dim E_{\lambda_1} \le m_1 \Rightarrow n-1 \le m_1$ (1)

$1 \le \dim E_{\lambda_2} \le m_2 \Rightarrow 1 \le m_2$ (2)

Know $f(t) = (\lambda_1 - t)^{m_1}(\lambda_2 - t)^{m_2} \underbrace{g(t)}$ → the presumably non-splittable part. $\deg(g) = x \ge 0$.

but $\deg(f(t)) = n \Rightarrow m_1 + m_2 + x = n$

From (1), (2), $m_1 + m_2 \ge (n-1) + 1 = n$

$\Rightarrow$ $x = 0$, so $g(t) = c \in \mathbb{F}$

So: $f(t) = c(\lambda_1 - t)^{m_1}(\lambda_2 - t)^{m_2}$, and thus f splits over $\mathbb{F}$.

But now: $m_1 + m_2 = n$, $m_1 \ge n-1$, $m_2 \ge 1$ $\Rightarrow$ only possibility is $m_1 = n-1$, $m_2 = 1$

So: $\dim(E_{\lambda_1}) = n-1$, $\dim(E_{\lambda_2}) = 1$

Hence A is diagonalizable

QED.