# Math 122 / Midterm 1 (Fall 2005)

**Due Wednesday, October 12 in class**

**1.** Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of homomorphisms of finite groups, with $f$ injective, $g$ surjective, and $\ker(g) = \mathrm{im}(f)$.

**(a)** Show that $\#B$ is the product of $\#A$ and $\#C$. Is there a stronger statement that you can make which implies this?

**(b)** Is $B$ isomorphic to the direct product of $A$ and $C$? If so, give a proof. If not, give a counter example.

**2.** **(a)** Define the center $Z$ of a group $G$, and show that it is a normal subgroup.

**(b)** If $G/Z$ is cyclic, show that $G$ is abelian.

**3.** Let $H$ be a subgroup of $G$ and, for an element $g \in G$, define the *double coset* $HgH = \{hgh' : h, h' \in H\}$.

**(a)** Show that the double cosets $HgH$ and $Hg'H$ for some $g, g' \in G$ are either equal, or have trivial intersection.

**(b)** If $H$ is normal, show that $HgH$ is equal to the left coset $gH$, for every $g \in G$.

**(c)** If $H$ is not normal, show that *some* $HgH$ is not equal to a left coset of $H$.

**(d)** Let $G = S_n$ and $H = S_{n-1}$ be the subgroup of permutations fixing the element $n$ of the set $\{1, 2, \ldots, n\}$ How many distinct double cosets of $H$ are there? How many elements are in each double coset?

**4.** Let $n$ be a positive integer, $S_n$ the permutation group on $n$ letters, and let $p$ be a prime.

**(a)** Show that if $n < p$ then $S_n$ contains no element of order $p$.

**(b)** Show that if $n \geq p$ then $S_n$ contains at least one element of order $p$.

**(c)** When is there an element of order $p$ in the subgroup $A_n$ of $S_n$?