Math 122 / Problem Set 1 Solution
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Problem 1

Solution 1: If $(I+A)$ is not invertible $\Leftrightarrow \ker(I+A) \neq \emptyset \Leftrightarrow \exists$ vector $\vec{x} \neq \vec{0}$ such that $(I+A)\vec{x} = 0 \Leftrightarrow A\vec{x} = -I\vec{x} = -\vec{x}$ for some $\vec{x} \neq \vec{0}$. Observe that $A^k\vec{x} = A^{k-1}(A\vec{x}) = A^{k-1}(-1)\vec{x}$. Continue factoring A^{k-1}, A^{k-2} and so on, we have $A^k\vec{x} = (-1)^k\vec{x}$. In any case, as $\vec{x} \neq \vec{0}$, this means that $A^k \neq 0$, which contradicts the condition of the problem. Therefore our initial assumption must not hold, and hence $(I+A)$ is invertible.

Solution 2: Consider $B = (I - A + A^2 - A^3 + \ldots + (-1)^k A^k)$. then $(I+A)B = I + (-1)^k A^{k+1} = I + 0$. Similarly, $B(I+A) = I$, thus B is the inverse of $(I+A)$. We have constructed a proper inverse, therefore $I+A$ is invertible.

Problem 2

(a) Let $A = (a_{ij})$ and $B = (b_{ij})$, then $A+B = (a_{ij} + b_{ij})$. Thus, tr $(A+B) = \Sigma_{i=1}^n (a_{ii} + b_{ii}) = \Sigma_{i=1}^n a_{ii} + \Sigma_{i=1}^n b_{ii}$ =tr A + tr B

(b) Using the same notation as above, $AB = (\Sigma_{k=1}^n a_{ik}b_{kj})$, and $BA = (\Sigma_{k=1}^n b_{ik}a_{kj})$.

Then tr $AB = \Sigma_{i=1}^n (\Sigma_{k=1}^n a_{ik}b_{ki}) = \Sigma_{i=1}^n (\Sigma_{k=1}^n b_{ki}a_{ik})$ =tr BA

(c) Assume that there exist matrics A, B such that $AB - BA = I$. Take the trace of both sides, we have $tr(AB - BA) = tr(I)$. But by part (a) and (b) tr$(AB - BA)$ =tr$(AB)-$tr$(BA) = 0$, so we have tr$(I) = 0$, which is never true as tr $I = n$, where n is the number of columns of the matrix. The contradiction shows that the equation has no real matrix solutions.

Problem 3

To show that G° is a group, we just need to show that it satisfies the group axioms as described in the textbook and the lectures.

- Identity: The element e which is the identity of G is the identity of G° because $\forall a \in G, e \circ a = a \cdot e = e \cdot a = a \circ e = a$

- Inverse : Since G is a group, every element $a \in G$ has a unique inverse a^{-1},and we show that this is also the unique inverse of a in $G^\circ : a \circ a^{-1} = a^{-1} \cdot a = e = a \cdot a^{-1} = a^{-1} \circ a$

- Closure: Given any $a, b \in G^\circ, a \circ b = b \cdot c, a \in G = G^\circ$ (equivalent of sets).

- Associativity: Given any $a, b, c, \in G^\circ, c \circ (b \circ a) = c \circ (a \cdot b) = (a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot (c \circ b) = (c \circ b) \circ a$

Problem 4

Solution 1 a. If both elements ab and ba have infinite order, then we are done. b. If at least one of them is finite, and without loss of generality

(WOLG), we let the order of ab be $nfinite$.

So we have $(ab)^n = e \rightarrow a(ba)^{n-1}b = e \rightarrow (ba)^{n-1} = a^{-1}b^{-1} \rightarrow (ba)(ba)^{n-1} = baa^{-1}b^{-1} = beb^{-1} = e \rightarrow (ba)^n = e$. So in this case, ba has finite order m, and $m|n, m \leq n$. However, by an entirely symmetric argument, we can prove that $(ab)^m = e$, which means that $n|m, n \leq m$. These two conditions stipulate then that $n = m$.

In all cases, then, the order of (ab) and (ba) are the same.

Solution 2: Notice that $b(ab)b^{-1} = ba$, therefore, ba is the image of ab in the conjugation by b. Conjugation, as Artin states, is an automorphism, therefore the order of the image and the preimage are the same. Thus order of ab is equal to order of ba.

Problem 5

We have to show that $\forall a, b \in G$, $ab = ba$. Now since all elements except e has order 2, we have $a^2 = b^2 = e$, or, $a = a^{-1}, b = b^{-1}$. Also $(ab) = (ab)^{-1} = b^{-1}a^{-1} = ba\ \forall a, b \in G$. q.e.d.