Math 122 / Problem Set 11

Written problems due Monday, December 12

Monday, December 5

1. Prove or disprove the following.
(a) The polynomial ring $\mathbb{R}[X, Y]$ in two variables is a Euclidean domain.
(b) The ring $\mathbb{Z}[X]$ is a principal ideal domain.

2. Factor 30 as product of prime elements in $\mathbb{Z}[i]$.

3. Let $R = \mathbb{Z}[\sqrt{-5}]$.
(a) Show that $R^\times = \{\pm 1\}$.
(b) Show that every element $r \in R$ can be written as a product of irreducible elements of R.
(c) Prove that $2, 3, 1 \pm \sqrt{-5}$ are irreducible elements of R.
(d) Using **(c)** show that R does not have unique factorization into irreducibles.

4. Show that $P = (p)$ is a nonzero prime ideal in a ring R (*cf.* last week's homework) if and only if p is a prime element.

Reading: Artin §§11.3, 11.4

Wednesday, December 7

5. Prove that two polynomials in $\mathbb{Z}[X]$ are relatively prime in $\mathbb{Q}[X]$ if and only if the ideal they generate in $\mathbb{Z}[X]$ contains a nonzero integer.

6. Prove that the following polynomials are irreducible in $\mathbb{Q}[X]$:
(a) $X^2 + 27X + 213$ **(b)** $X^3 + 6X + 12$ **(c)** $8X^3 - 6X + 1$

7. Let $f(X) = X^n + c_{n-1}X^{n-1} + \cdots + c_1X + c_0$ be a monic polynomial in $\mathbb{Z}[X]$, and let $r \in \mathbb{Q}$ be a rational root of $f(X)$. Prove that r is an integer.

Reading: Artin §§12.1, 12.2

Friday, December 9

8. **(a)** Let $\varphi : V \to W$ be a homomorphism of modules over a ring R and let V', W' be submodules of V, W respectively. Prove that $\varphi(V')$ is a submodule of W and that $\phi^{-1}(W')$ is a submodule of V. What does this say about $\ker \varphi$ and $\mathrm{im}\varphi$?

A module is called *simple* if it is not the zero module and if it has no proper submodule.

(b) Prove "Schur's Lemma": Let $\varphi : V \to W$ be a homomorphism of simple modules. Then either φ is zero, or else it is an isomorphism.

(c) Prove that any simple module V over R is isomorphic to R/M, where M is a maximal ideal.

9. Let R be a ring, and let V be a free R-module of finite rank. Prove or disprove the following:

(a) Every set of generators contains a basis.

(b) Every linearly independent set can be extended to a basis.

Reading: Artin §§12.3, 12.4