Math 122 / Problem Set 4

Written problems due Monday, October 17

Wednesday, October 12

1. A complex $n \times n$ matrix is called *hermitian* if $a_{ij} = \overline{a_{ji}}$ for all i, j. Show that the hermitian matrices form a real vector space. Find a basis for that space and determine the dimension as a function of n.

2. Let W be a subspace of a finite dimensional vectorspace V.

(a) Prove that there is a subspace U of V such that $U + W = V$ and $U \cap W = \{0\}$.

(b) Prove that there is no subspace U such that $W \cap U = \{0\}$ and $\dim W + \dim U > \dim V$.

3. Prove that a square matrix is invertible if and only if its columns are linearly independent.

4. **(a)** Let p be a prime. Count the number of ordered bases $\{e_1, e_2\}$ of a 2-dimensional vector space over $\mathbb{Z}/p\mathbb{Z}$.

(b) Show each basis uniquely determines an invertible matrix in $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$, and determine the order of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$.

(c) Determine the order of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$.

(d) Find an element of order p in $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$.

Reading: Artin §§4.1, 4.2

Friday, October 14

5. Let V be the vectorspace of polynomials of degree ≤ 3 over $\mathbb{R}$. Let $T : V \to V$ be the linear operator mapping

$$T(P(X)) = \frac{d^2}{dX^2} P(X).$$

Write the matrix for T with respect to the basis $\{1, X, X^2, X^3\}$. What is its rank?

6. Prove that the space $\mathrm{M}_n(\mathbb{R})$ of all $n \times n$ real matrices is the direct sum of the spaces of symmetric matrices ($A = {}^tA$) and of skew symmetric matrices ($A = -{}^tA$).

7. When the field of scalars is $\mathbb{Z}/p\mathbb{Z}$, finite-dimensional vectorspaces have finitely many elements. In this case, Artin's formulas 4.1.6 and 2.6.15 both apply. Reconcile them.

8. Let A be an $m \times n$ matrix over a field F. Prove that the space of solutions of the linear system $AX = 0$ ($X \in F^n$) has dimension at least $n - m$.

Reading: Artin §§4.3, 4.4