Math 122 / Problem Set 5

Written problems due Monday, October 24

Monday, October 17

1. An operator on a vector space V is called *nilpotent* if $T^k = 0$ for some k. Let T be a nilpotent operator, and let $W_i = \text{image}(T^i)$.

(a) Prove that if $W_i \neq \{0\}$ then $\dim(W_{i+1}) < \dim(W_i)$.

(b) Prove that if V is a space of dimension n and T is nilpotent, then $T^n = 0$.

2. **(a)** Let T be a linear operator having two linearly independent eigenvectors with the same eigenvalue λ. Is it true that λ is a multiple root of the characteristic polynomial of T?

(b) Suppose that λ is a multiple root of the characteristic polynomial. Does T have two linearly independent eigenvectors with eigenvalue λ?

3. Do A and ${}^tA$ have the same eigenvalues? The same eigenvectors?

4. Let A be a complex matrix such that $A^n = I$. Prove that the eigenvalues of A are nth roots of unity, i.e. powers of $\zeta_n = e^{2\pi i/n}$.

Reading: Artin §§4.5

Wednesday, October 19

5. Prove that every orthonormal set of n vectors in $\mathbb{R}^n$ is a basis.

6. Prove that O_n and SO_n are subgroups of $\mathrm{GL}_n(\mathbb{R})$, and determine the index of SO_n in O_n.

7. Let v be a vector of unit length, and let P be the plane in $\mathbb{R}^3$ orthogonal to v. Describe a bijective correspondence between points on the unit circle in P and matrices $A \in SO_3$ whose first column is v.

Reading: Artin §§5.1, 5.2

Friday, October 21

8. Let G be a finite group of rotations of the plane about the origin. Prove that G is cyclic.

9. Recall that a *rigid motion* is a map $m : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$|m(X) - m(Y)| = |X - Y|$$

for all $X, Y \in \mathbb{R}^n$. Prove from the definition that a rigid motion is a bijection.

10. Prove that a linear operator on $\mathbb{R}^2$ is a reflection if and only if its eigenvalues are 1 and -1 and its eigenvectors are orthogonal.

Reading: Artin §§5.3, 5.4