Math 122 Wednesday, October 19

$V = \mathbb{R}^n$ $e_1 = (1,0,\dots,0)$, $e_2 = (0,1,0,\dots,0)$, $\vdots$, $e_n = (0,\dots,0,1)$

$\sum c_i e_i = (c_1,\dots,c_n) \in \mathbb{R}^n$

$T: V \to V$ one-to-one and onto $\longleftrightarrow$ A with $\det A \neq 0$

Goal: Put more structure on V and look at the subgroup $H \subset G = GL_n(\mathbb{R})$ preserving that structure.

Namely, a positive definite inner product on V: $\langle\, , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$\langle (v_1,\dots,v_n),(w_1,\dots,w_n)\rangle = v_1w_1 + v_2w_2 + \dots + v_nw_n = \sum v_iw_i \quad \text{(dot product)}$$

defn an orthonormal basis wrt $\langle\ \rangle$ is a basis $e_1,\dots,e_n$ of V such that $\langle e_i,e_j\rangle = 0$ if $i \neq j$ and $\langle e_i,e_i\rangle = 1$

If you fix v the map $\mathbb{R}^n \to \mathbb{R}$ $w \mapsto \langle v,w\rangle$ is linear
" " w " " " " $v \mapsto \langle v,w\rangle$ is linear

$\}$ $(v,w) \mapsto \langle v,w\rangle$ is a bilinear transformation

Write $v \perp w$ if $\langle v,w\rangle = 0$. Note $\langle v,v\rangle = \sum v_i^2 \geq 0$ with equality iff $v=0$ in $\mathbb{R}^n$

More generally if $\langle\ \rangle : V \times V \to \mathbb{R}$ a bilinear map where V an n-dimensional vector space over $\mathbb{R}$, and with $\langle v,v\rangle \geq 0$ $\forall v$ with equality iff $v=0$.

Then $\exists$ an orthonormal basis $\langle e_1,\dots,e_n\rangle$ for V.

Proof with Gram-Schmidt

$\langle v_1,\dots,v_n\rangle \rightsquigarrow e_1 = \frac{v_1}{\sqrt{\langle v_1,v_1\rangle}}$ so $\langle e_1,e_1\rangle = 1$

$v_2' = v_2 - \langle v_2,e_1\rangle e_1$ so $\langle v_2',e_1\rangle = 0$

$e_2 = \frac{v_2'}{\sqrt{\langle v_2',v_2'\rangle}}$ so $\langle e_2,e_2\rangle = 0$

This gives an isomorphism $V \xrightarrow{\sim} \mathbb{R}^n$ $v = \sum c_ie_i \mapsto (c_1,\dots,c_n)$ so that the inner product on V maps to the standard dot product on $\mathbb{R}^n$! So $\langle \sum c_ie_i, \sum d_ie_i\rangle = \sum \langle c_ie_i, d_ie_i\rangle = \sum c_id_i$. Hence it suffices to only consider the standard dot product.

Now define the orthogonal group $O(n) \subset GL(V)$, $V = \mathbb{R}^n$, to be those $T: V \to V$ which satisfy $\langle Tv,Tw\rangle = \langle v,w\rangle$ for all $v,w \in V$. So $O(n)$ is the set of maps that preserve the inner product. Clearly closed under multiplication = composition of transformations.

Write $A = \begin{pmatrix} & | & \\ \cdots & Te_i & \cdots \\ & | & \end{pmatrix}$ Then T is orthogonal iff the Te_i are orthonormal (assuming $e_1, \ldots, e_n$ was an orthonormal basis). By matrix multiplication, this is true iff ${}^tA \cdot A = I$.

$A = (a_{ij})$ ${}^tA := (a_{ji})$

Moreover if $A^t A = I$ then the Te_i are orthonormal $\Longrightarrow$ T is an orthogonal transformation

$O(n) = \{ A \in Gl_n(\mathbb{R}) \mid {}^tA \cdot A = I \}$ Note $\det A = \det A^t \Rightarrow (\det A)^2 = 1 \Rightarrow \det A = \pm 1$

ex. $\langle \pm 1 \rangle = \underset{a \cdot a = 1}{O(1)} \subset \underset{A = (a)}{Gl_1(\mathbb{R})} = \mathbb{R}^* = \mathbb{R} - \{0\}$

$v = \sum c_i e_i$, $w = \sum d_i e_i$ $\quad \sum c_i d_i = \langle v, w \rangle$, $\langle Tv, Tw \rangle \overset{\text{linearity of } T}{=} \langle \sum c_i Te_i, \sum d_i Te_i \rangle \overset{\text{by bilinearity + fact } Te_i \text{ orthonormal}}{=} \sum c_i d_i = \langle v, w \rangle$

More generally 1) $\det A = \pm 1$ (note $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in O(n) \Rightarrow$ can get $\det A = -1$)
2) any eigenvalue c of T is ± 1. Suppose $Tv = cv$, $v \neq 0$.
Then $\langle v, v \rangle = \langle Tv, Tv \rangle = \langle cv, cv \rangle = c^2 \langle v, v \rangle \Rightarrow 1 = c^2$ as $\langle v, v \rangle \neq 0 \Rightarrow c = \pm 1$

$O(n) \xrightarrow{\det} \langle \pm 1 \rangle \subset \mathbb{R}^\times$ Kernel $= SO(n) = \{ A \mid {}^tA \cdot A = I, \det A = 1 \}$ the special orthogonal group. This subgroup is normal and has index 2.

Consider $O(2) \overset{2}{\supseteq} SO(2)$ If $v = (x, y)$ then $\langle v, v \rangle = x^2 + y^2 \Rightarrow |v| = \sqrt{\langle v, v \rangle} = \sqrt{x^2 + y^2}$ with the positive square root = length of v. Define $|v - w|$ to be the distance from the endpoint of v to the endpoint of w. Also $\langle v, w \rangle = \cos\theta \, |v| \, |w|$ where θ is the angle between the vectors. Note $\langle v, w \rangle = 0 \Longleftrightarrow \theta = \frac{\pi}{2} \Rightarrow v \perp w$.

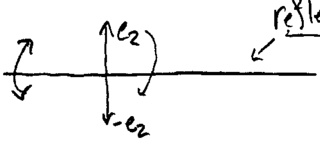
$T \in SO(2)$ preserves $\langle \; \rangle$ so preserves length. Hence $Te_1 = (\cos\theta, \sin\theta)$ $0 \leq \theta < 2\pi$
Must have $Te_1 \perp Te_2 \Rightarrow Te_2 = (-\sin\theta, \cos\theta)$ or $(\sin\theta, -\cos\theta)$. Which one?
Check $\det T = 1 \Longrightarrow T$ is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = A_\theta$ $\quad \det A_\theta = \cos^2\theta + \sin^2\theta = 1$.

Note $A_\theta \cdot A_{\theta'} = A_{\theta + \theta'} = A_{\theta'} \cdot A_\theta \Rightarrow SO(2)$ is abelian.

Have isomorphism $SO(2) \xrightarrow{\sim} U = \{ z \in \mathbb{C}^* : |z| = 1 \}$
$A_\theta \longmapsto z = e^{i\theta}$

For the other coset of $SO(2)$ in $O(2)$ take $B_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ the other choice of sign.
Take representative $B_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in O(2) - SO(2)$. Note $B_\theta = A_\theta \cdot B_0$.

What is B_0 geometrically?



reflection in this line

$(x, y) \mapsto (x, -y)$

All $B_\theta \in O(2) - SO(2)$ are reflections in some line through the origin. $B_\theta^2 = I$.

$B_0 A_\theta B_0 = A_{-\theta}$ in particular, $O(2)$ is NOT abelian