Math 122

10/20/2005

Review V: vectorspace

$T: V \to V$ is orthogonal if it preserves $\langle , \rangle$

$\langle Tv, Tw \rangle = \langle v, w \rangle \quad \forall v, w \in V$

T preserves length $\|v\| = \sqrt{\langle v,v \rangle}$

T. preserve angles $c^2 = a^2 + b^2 - 2ab\cos\theta$.

Write T as a matrix A wrt to a basis (orthonormal)

$x, y \in \mathbb{R}^n \quad \langle x,y \rangle = \sum x_i y_i = {}^t x \, y$.

Propositions: The following are equivalent

(i) ${}^t A \cdot A = I$

(ii) $\langle Ax, Ay \rangle = \langle x, y \rangle \quad \forall x, y \in \mathbb{R}^n$

(iii) The columns of A form an ONB.

Pf) (i ⇒ ii) $\langle Ax, Ay \rangle = {}^t(Ax) \, Ay = {}^t x \, {}^t A A y = {}^t x \, I y$
$= {}^t x y = \langle x, y \rangle$

(ii ⇒ iii) $x = e_i, y = e_j \; \langle A_i, A_j \rangle = \langle e_i, e_j \rangle = 0$ if $i \neq j$
$= 1$. $i = j$

(iii → i)

${}^t A A = (b_{ij}) \to (b_{ij}) = \langle Ae_i, Ae_j \rangle$

Now, consider SO_3.

Claim: $A \in SO_3$ has at least 1 eigenvalue equal to 1

Pf. Characteristic poly $f(x)$ of degree 3 and real coefficients.

Say $f(z) = 0$ then $f(\bar{z}) = \bar{z}^3 - a_2\bar{z}^2 + a_1\bar{z} - a_0$
$= \overline{f(z)} = 0$

⇒ f has a real root $c \Rightarrow \exists v : Tv = cv$.

then $\langle v, v \rangle = \langle cv, cv \rangle \; (= \langle Tv, Tv \rangle)$
$= c^2 \langle v, v \rangle \Rightarrow c^2 = 1 \Rightarrow$

Now

But products of roots of $f = 1$ $\rightarrow$ 2 complex roots, 1 real root
$\rightarrow$ 3 real roots.

$$1 = \begin{cases} c\, z \cdot \bar{z} = c|z| \Rightarrow c = +1 \\ c \cdot (\pm 1)(\pm 1) \Rightarrow \text{at least one is } +1 \end{cases}$$

How do we the write SO_3 out explicitly?

Write $A \in SO_3$ wrt to an onb containing the eigenvector for eigenvalue 1

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{B}} \\ 0 & & \end{bmatrix}$$

(Check the canonical forms to understand why A decomposes into 2 blocks)

Claim that $B \in SO_2$, $\det A = \det B = 1$ (by calculating $\det A$)

$$\det A = (1)\det(B) - 0 + 0 = \det(B)$$

In fact, can write $B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

<u>Group of Rigid Motions</u>

A rigid motion also called an isometry is a map
$m: \mathbb{R}^n \to \mathbb{R}^n$ s.t. $|mX - mY| = |X - Y|$ $\forall x, y \in \mathbb{R}$

ex! Any orthogonal $T: \mathbb{R}^n \to \mathbb{R}^n$ is a Rigid motion

ex 2: Translation $m: x \to x + b$
$$|x + b - (y + b)| = |x - y|$$

Claim: $M = \{\text{rigid motion}\}$ is a group

Proposition

A rigid motion fixes the origin $\Leftrightarrow$ it is orthogonal

Pf: $\Leftarrow$ orthogonal $\Rightarrow$ preserves length (by $\langle , \rangle$)

$\Rightarrow$: We know $|mX - mY| = |X - Y|$
thus
$$\langle mX - mY, mX - mY \rangle = \langle X - Y, X - Y \rangle$$
Let $y = 0 \rightarrow$ $\langle mX, mX \rangle = \langle X, X \rangle$

So now consider $\langle X - Y, X - Y \rangle = \langle X, X \rangle - 2\langle X, Y \rangle + \langle Y, Y \rangle$
$\langle mX - mY, mY - mY \rangle = \langle mX, mX \rangle - 2\langle mX, mY \rangle + 2(mY, mY)$

$\Rightarrow$ $\langle x, y \rangle = \langle mX, mY \rangle \Rightarrow m \in O(n)$

<u>Proposition</u>: Any rigid motion is the composition of something in $O(n)$ with a translation.

Pf: $m \in M$, $m(0) = b \in \mathbb{R}^n$.
let
$t_{-b}: x \to x - b$.

$\rightarrow t_{-b} \circ m$ fixes $0 \Rightarrow t_{-b} \circ m \in O(n)$
$\Rightarrow m = t_b \circ A$. #

Symmetries of the plane (consider linear)

<u>Define</u> $m \in M$ is orientation preserving if $\det m = 1$,
orientation reversing if $\det m = -1$
If m is not linear, then just consider geometric interpretation.

<u>Examples</u>. translation, reflection, rotation, glide-reflection
glide reflection = reflection about l, translation along l.

Theorem: All rigid motions of the plane have one of these forms

Translation, Rotation: Orientation preserving

Reflection, glide reflection: orientation reversing

Proof: $m \in M =$ group of rigid motions of the plane then

$$m = t_a \cdot A_\theta \cdot r^i$$

$t_a : x \to x + a$

A_θ : rotate by θ

r : reflection about axis

because an element in O_2 lies either in SO_2 or rSO_2 (the other coset)

Further such expressions are unique, otherwise:

$$t_a A_\theta = t_b \cdot A_\phi \quad \text{(the orientation-preserving part)}$$

$$\Rightarrow \underbrace{t_{-b} \cdot t_a}_{\text{Translation}} = \underbrace{A_\phi A_{-\theta}}_{\text{Rotation}}$$

Translation = Rotation only when both = 0.

$\Rightarrow$ $t_a = t_b$

$A_\theta = A_\phi$

Say m is orientation preserving $\Rightarrow m = t_a \circ A_\theta$

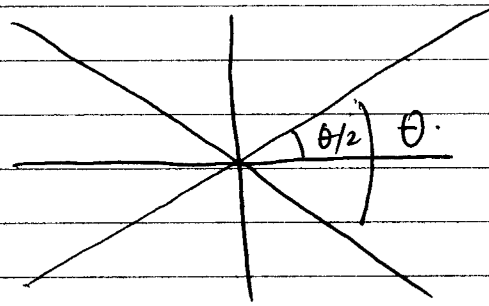
m fixes a point in the place (find it w/ a geometric argument)

So $m(p) = p$.

$$m(p+x) = t_a \circ A_\theta (p+x)$$
$$= A_\theta(p+x) + a$$
$$= A_\theta p + A_\theta x + a$$
$$= (t_a A_\theta)(p) + A_\theta(x)$$
$$= A_\theta(x)$$

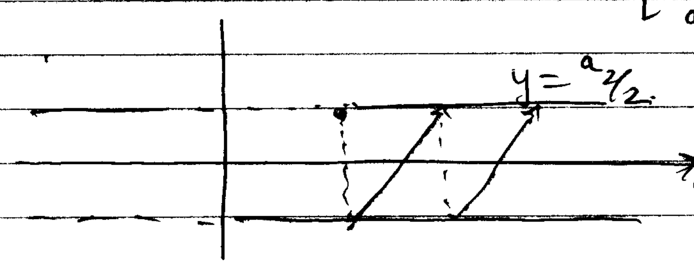
$\Rightarrow$ m is rotation around p.

m is orientation reversing $m = t_a A_\theta r$.



$A_\theta r = r'$: reflection about a line through the origin
Change coordinates ~~so that~~,

Now $m = t_a \circ r$ let $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$



→ It is a glide reflection along the axis of reflection because at each time, $m = t_a \circ r$, moves a point along the axis and also reflects it.

<u>Notes</u>: two subgroups of M:

1) $O \subset M$ orthogonal
$T \subset M$ transformation $T \simeq \mathbb{R}^n$.

2) The group of rigid motions that fix p conjugate to $O(n)$
by. $O(n)$ t_{-p}

3) $\varphi : M \to O$ is a homomorphism with kernel T
$t_a \circ A_\theta \circ r \to A_\theta \circ r \to$

Exercise:
$T \triangleleft M$: