Math 122 Monday, October 24

$M(n)$ group of motions of $\mathbb{R}^n$ $m:\mathbb{R}^n\to\mathbb{R}^n$ bijection, preserves $|x-y|=|mx-my|$ $\forall x,y$

Ex $m(v)=v+b$ $b\in\mathbb{R}^n$ translations (formally denoted $t_b(v)$), subgroup $\mathbb{R}^n\subset M(n)$
$m(v)=Av$ orthogonal transformation $A\in O(n)$ gives a subgroup $O(n)\subset M(n)$ that preserves $0\in\mathbb{R}^n$
$O(n)\cap\mathbb{R}^n=\{e\}$ as nothing else in $\mathbb{R}^n$ preserves 0.

Prop 1) The subgroup of $M(n)$ fixing 0 is equal to $O(n)$.
2) Any motion m has the form $m(v)=Av+b$ where $b=m(0)$.

As a set $M(n)=O(n)\times\mathbb{R}^n$ but this is not true as a group (not a direct product)

$(A,b)\cdot(A',b')(v)=(A,b)(A'v+b')=A(A'v+b')+b=AA'v+(Ab'+b)=(AA',Ab'+b)$.

Note there is a group homomorphism $f:M(n)\to O(n)$ that is surjective
$m=(A,b)\longmapsto A$
note this is a homomorphism by the above computation. $\text{kernel}=\{(I,b)\mid b\in\mathbb{R}^n\}\cong\mathbb{R}^n$ the translations
So $\mathbb{R}^n\triangleleft M(n)$ and by 1st Isomorphism Thm $M(n)/\mathbb{R}^n\cong O(n)$.

Can see that $(I,b)\,O(n)\,(I,b)^{-1}$ is the subgroup of $M(n)$ that preserve the vector p

M(2) and Euclidean Geometry

Claim The triangles ABC and $A'B'C'$ in $\mathbb{R}^2$ are congruent $\iff$ $\exists\, m\in M(2)$ with $m(A)=A'$, $m(B)=B'$, and $m(C)=C'$, i.e. the triangles are permuted by a motion

Idea due to Felix Klein studying geometry on $S^2\subset\mathbb{R}^3 \longleftrightarrow G=SO_3$.

Pf: If $m(ABC)=A'B'C'$ the triangles are congruent by side-side-side. Now assume ABC and $A'B'C'$ are congruent – use translation by $b=A'-A$ to map A to A' (then for convenience, $b=-A'$ to map both points to 0). Now rotate by $\mathrm{rot}(\theta)\in SO(2)$ to map B to B', possible as these points have the same distance from the origin. Now if $C\neq C'$ use a reflection in $O(2)$ through the line AB.

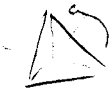
Is the motion $m(ABC)=A'B'C'$ unique? What is the subgroup $\Gamma\subset M(2)$ fixing the ΔABC?

Note if m fixes ABC then it fixes the medians from each side so it fixes the point where they intersect:

fixed point

[Aside: note if g fixes a set/line/whatever S that means $g(S)\subset S$ not each point $s\in S$ is mapped to $g(s)=s$. If a point p is fixed however, $g(p)=p$.]

If ΔABC is equilateral, $\Gamma \cong S_3$ given by $120°$ rotation and 3 reflections



If ΔABC is isosceles, $\Gamma \cong S_2$ given by one reflection

If ΔABC is not then Γ is trivial.

Finite subgroups of $M(n)$. Assume $\Gamma \subset M(n)$ is finite.

Claim A conjugate of Γ is a finite subgroup of $O(n)$.

Pf) Let v be any vector in $\mathbb{R}^n$. Then $p = \frac{1}{\#\Gamma} \sum_{g \in \Gamma} g(v)$ is fixed by all $h \in \Gamma$ (proof next time).
Then $\Gamma \subset t_p \cdot O(n) t_p^{-1}$ the subgroup of $M(n)$ fixing p. So $t_p^{-1} \Gamma t_p < O(n)$.

$\Gamma \subset O(2)$. What does Γ look like?

Case 1) $\Gamma \subset SO(2) \Longrightarrow \Gamma$ is cyclic of any order n generated by $\mathrm{rot}(\frac{2\pi}{n}) = g$, $\Gamma = \langle e, g, \dots, g^{n-1} \rangle$

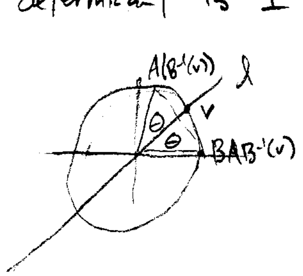
Case 2) $\Gamma \subset O(2)$ but not $SO(2) \Longrightarrow \Gamma$ is dihedral of order $2n$ $\quad rgr = g^{-1}$, $g^n = e$, $r^2 = e$

$\Gamma = \langle e, g, g^2, \dots, g^{n-1}, r, rg, \dots, rg^{n-1} \rangle$ $\quad r$ = reflection

Name comes from the dihedron, a polygon embedded in the plane bisecting a sphere.

Pf: Say $\Gamma \subset SO(2)$. Let $g \in \Gamma$ be something of the maximal order n. Then $g = \mathrm{rot}(\theta)$ as $g \in SO(2)$ and $\theta = \frac{2\pi k}{n}$. Must have $\gcd(k,n) = 1$ or order of $g < n$. Also as $\gcd(k,n) = 1$ $\exists$ h such that $h \cdot k \equiv 1 \pmod n \Longrightarrow g^h = \mathrm{rot}(\frac{2\pi}{n})$. Now let $g \in \Gamma$ be rotation by the smallest angle $\theta = \frac{2\pi}{n}$ (note $\theta = \frac{2\pi k}{n} \Longrightarrow \theta = \frac{2\pi}{n}$ in Γ by above which is smaller). So every $h \in \Gamma$ is a power of g.

Now say $\Gamma \not\subset SO(2)$. Then $\Gamma \cap SO(2) = \Gamma_0 \triangleleft \Gamma$ which is cyclic by what we just showed. Let $r \in \Gamma \setminus \Gamma_0 \Longrightarrow$ non-trivial coset $r\Gamma_0 = \langle r, rg, \dots, rg^{n-1} \rangle$. Remains to show that $rgr^{-1} = g^{-1}$ in Γ for any reflection $r \in \Gamma - \Gamma_0$ Why? This is true for any reflection and rotation in $O(2)$.
Let A be a rotation and B be a reflection, then BAB^{-1} is a rotation because its determinant is 1 so it's in $SO(2)$. Pick some point v on the line l fixed by B.



$B^{-1}(v) = v$

Then B^{-1} fixes this point, AB^{-1} rotates v by θ and BAB^{-1} reflects this through l, giving a net rotation by $-\theta = A^{-1}$. So $BAB^{-1} = A^{-1}$.