10/05/2005

Problem set 3. Due Wednesday.

Product of Groups.

Definition If G & G' are groups then $G \times G = \{g, g' : g \in G, g \in G'\}$

Claim - $G \times G'$ is a group with $(g,g')\cdot(h,h') = (gh, g'h')$
- $(e \times e')$ is the identity
- Associativity, closure : obvious

$$(g,g')^{-1} = (g^{-1}, g'^{-1})$$
$$(g,g')\cdot(g^{-1}, g'^{-1}) = (e, e')$$

Example $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$\Rightarrow \{(0,0), (0,1), (1,0), (1,1)\}$

$$(0,1)^2 = (0,1)(0,1) = (0,0)$$
$$(1,0)^2 = (0,0)$$
$$(1,1)^2 = (0,0)$$
$$(0,0)^2 = (0,0)$$
$$(0,1)(1,0) = (1,1)$$

$\Rightarrow$ The Klein-4 group. (each non-identity element has order 2)

In contrast: $(\mathbb{Z}/2\mathbb{Z})(\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

+ $\exists$ two inclusion homomorphisms:

$$G \hookrightarrow G \times G' \qquad g \mapsto (g, e')$$
$$G' \hookrightarrow G \times G' \qquad g' \mapsto (e, g')$$
$$G \cong G \times 1 \subset G \times G'$$
$$G' \cong 1 \times G' \subset G \times G'$$

projection homomorphisms

$$G \times G' \longrightarrow G \; : \; (g, g') \longmapsto g$$
$$G \times G' \longrightarrow G' \quad\;\; (g, g') \longmapsto g'$$

* H, K: subsets of a group.
set product $HK = \{ hk \mid h \in H, \; k \in K \}$.

example $G = S_3$
$H = \{ e, (12) \}$ $(1$
$\Rightarrow$
So: • $((13)H)H = (13)(H \cdot H) = (13)H \cdot \; (= \{(13), (13)(12$
• $(H)(13)H = \{ e(13), e(123), (12)(13) = (132),$
$(12)(123) = (23) \}$.

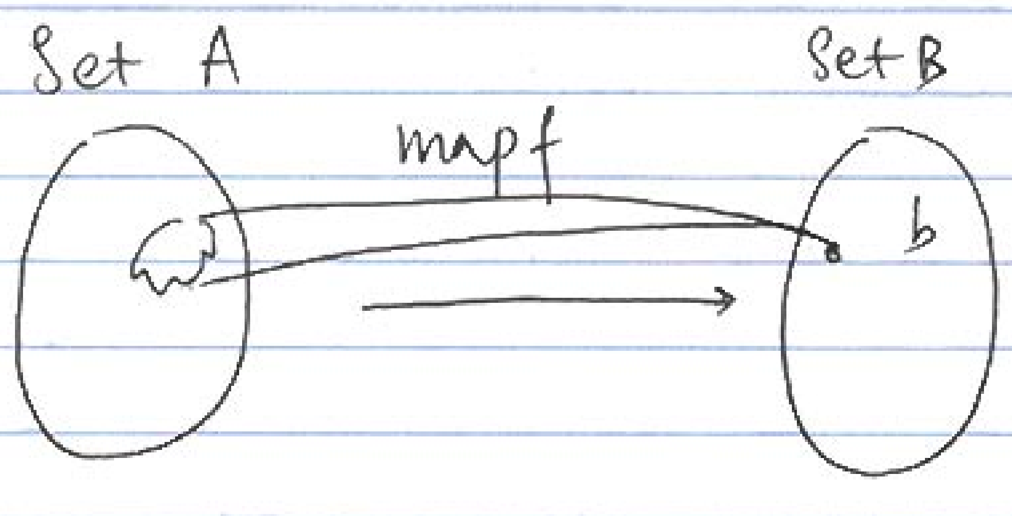
$\Rightarrow$ Not the same.

Recall

$H \triangleleft G$: $ghg^{-1} \in H \quad \forall h \in H \Rightarrow gHg^{-1} = H \rightarrow gH = Hg$.

$$(aH)(bH) = (Ha)(bH) = H(ab)H = abHH = abH$$

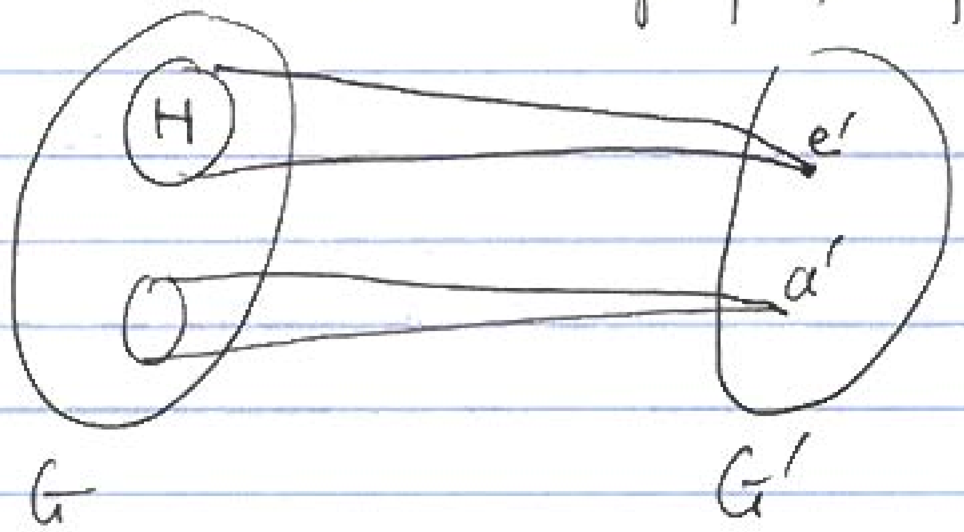
A slightly different perspective



Definition the fibres of f are subsets of A of the form $f^{-1}(b)$
for $b \in B$

If f is well-defined, fibres of f partition A

When G, G' are groups, f is a homomorphism



by definition $\ker f = f^{-1}(e') = H$.

Claim: the fibres of f are the cosets of $H = \ker f$

$$aH = f^{-1}(f(a))$$

Proof: let $ah \in aH$ then $f(ah) = f(a) \cdot f(h) =$

$$= f(a)e'$$
$$= f(a) \Rightarrow aH \subset f^{-1}(f(a))$$

$$b \in f^{-1}(f(a)) \qquad f(b) = f(a)$$
$$\Rightarrow \quad f(b) f(a)^{-1} = f(a) f(a)^{-1} = e'$$
$$f(b) f(a^{-1}) = e' = f(ba^{-1})$$
$$\Rightarrow \quad ba^{-1} \in H \Leftrightarrow \quad b \in aH.$$

* The correspondance between the image group & the fibres (coset)'

* Canonical Projection Map

$$H \triangleleft G \Rightarrow f: \quad G \to G/H$$
$$g \to gH$$
$$f(ab) = abH = aHbH = f(a)f(b)$$
and $H = \ker f$.

First Isomorphism Theorem

Given a group homomorphism $f: G \to G'$ with kernel H, th

$G/H \simeq \operatorname{im} f$ by the map $\bar{f}(aH) = f(a)$

Or, the following diagram is commutative:

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "c"'] & & G' \arrow[dl, "\bar{f}"] \\ & G/H \arrow[ur] & \end{tikzcd}$$

$f = \bar{f} \circ c$

$a \in G$: $\bar{f} \circ c(a) = \bar{f}(aH)$
$= f(a).$

Proof.

① $\bar{f}$ is well defined because given $aH = bH$, $\bar{f}(aH) = \bar{f}(b$

$\bar{f}(aH) = f(a)$

$\bar{f}(bH) = f(b)$

Since $aH = bH \Rightarrow ab^{-1} \in H \Rightarrow f(ab^{-1}) = e \Rightarrow f(a)f(b^{-1}) = e$

$\Rightarrow f(a)f(b^{-1}) = f(a)(f(b))^{-1} = e$

$f(a) = f(b)$

$\Rightarrow \bar{f}(aH) = \bar{f}(bH)$

2) $\bar{f}$ is a hom because $\bar{f}(aH)\bar{f}(bH) = f(a)f(b) = f(ab$

$= \bar{f}(abH)$

$= \bar{f}(aHbH)$.

Bijection by above argument (1).

Examine the structure of some quotient groups:

<u>example 1</u> $GL(\mathbb{R}) / SL_n(\mathbb{R})$

$\det : GL(\mathbb{R}) \longrightarrow \mathbb{R}^\times$

$\det(AB) = \det A \cdot \det B \rightarrow \det$ is a hom.

so: check that

1) $\ker(\det) = SL_n(\mathbb{R})$

2) det is surjective: indeed, as $\det\begin{pmatrix} a & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = a$

$\Rightarrow GL_n(\mathbb{R}) / SL_n(\mathbb{R}) = R^\times.$

<u>example 2</u>: S_n / A_n

$$\begin{tikzcd} S_n \arrow[r] \arrow[rr, bend right, "sign"'] & GL_n(\mathbb{R}) \arrow[r, "\det"] & R^* \end{tikzcd}$$

$A_n = \ker(\text{sign})$ (permutation of sign 1).

$\Rightarrow S_n / A_n \simeq \langle \pm 1 \rangle.$

Sketch of proof: (ab) has det -1.
and all permutation is a product of 2cycles.

<u>example 3</u>: $\mathbb{C}^\times / \mathcal{U}$

$\mathcal{U} =$ unit circle

$= \{ z \in \mathbb{C}^\times \mid |z| = 1 \}.$

$= \{ re^{i\theta} \in \mathbb{C}^\times \mid r = 1 \}.$

$\therefore f : \mathbb{C}^\times \longrightarrow \mathbb{R}^\times$

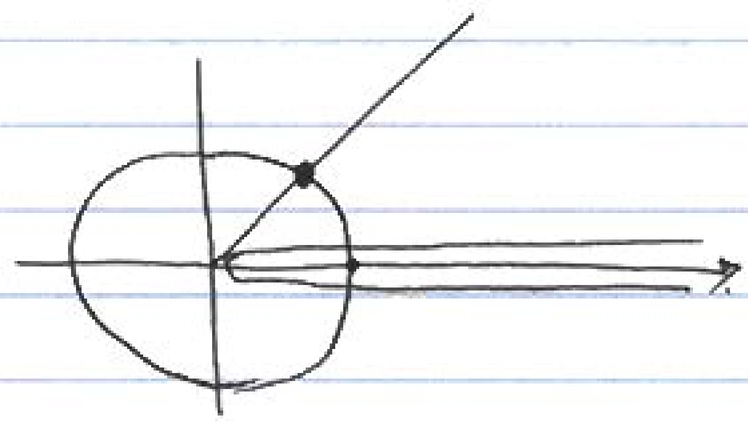
$z \longmapsto |z|$

$\ker f = \mathcal{U}$

$\operatorname{im} f = \mathbb{R}^\times_{>0}.$

$\Rightarrow \mathbb{C}^\times / \mathcal{U} \simeq \mathbb{R}^\times_{>0}.$

<u>example 4</u>.



(the other way round)

$\mathbb{C}^\times / \mathbb{R}^\times_{>0}$

$z = re^{i\theta}, r > 0, \quad \theta \in [0, 2\pi)$

$\therefore \mathbb{C}^\times \longrightarrow \mathcal{U}$

$z \longmapsto e^{i\theta}$

$z \longmapsto \frac{z}{|z|}$

$\Rightarrow \mathbb{C}^\times / \mathbb{R}_{>0} \simeq \mathcal{U}.$

<u>example 5</u>.

$\mathbb{R}^+ / \mathbb{Z} \simeq \mathbb{R}^+ / 2\pi\mathbb{Z} \simeq G?$

$f \; \mathbb{R}^+ \longrightarrow \mathcal{U}$

$x \longmapsto e^{ix}$

$\ker f = 2\pi\mathbb{Z}$

$\mathbb{R}^+ / 2\pi\mathbb{Z} \simeq \mathcal{U}.$

$g : \mathbb{R}^+ \to \mathcal{U}$

$x \longmapsto e^{2\pi i x}$

$\ker g = \mathbb{Z}.$