Math 122 Monday, September 26

Recall a homomorphism of groups is a map $G \xrightarrow{f} G'$ such that $\forall g_1, g_2 \in G$ $f(g_1 g_2) = f(g_1) f(g_2)$.

If $G \xrightarrow{f} G'$ and $G' \xrightarrow{h} G''$ are homomorphisms then so is $h \circ f$ because $h \circ f(g_1 g_2) = h(f(g_1) \cdot f(g_2)) = h \circ f(g_1) \cdot h \circ f(g_2)$.

A homomorphism is an isomorphism if it is a bijection of sets. In this case $f^{-1} : G' \to G$ is also a homomorphism. A homomorphism is an automorphism if f is an isomorphism from G to itself $G \xrightarrow{f} G$.

ex: $G \xrightarrow{f} G$ $f(g) = g^{-1}$ is a bijection of sets. $f(gh) = (gh)^{-1} = h^{-1} g^{-1} \overset{?}{=} f(g) f(h) = g^{-1} h^{-1}$
f is a homomorphism $\iff$ G is abelian

Note $Aut(G)$ is a subgroup of $Aut(T)$ where T is the set of elements in G.

Ex $G = \{e, g\}$ $Aut\, G = \{id\}$ because only bijective homomorphism preserves the identity and hence also g.

Ex $G = \{e, g, g^2\}$ $Aut(G)$ of order $2 = \left\{ id ; \begin{matrix} e \mapsto e \\ g \mapsto g^2 \\ g^2 \mapsto g \end{matrix} \text{ the inverse map} \right\}$

Of course $Aut(T) \simeq S_3$ so $Aut(G) \subsetneq Aut(T)$.

Construct a group homomorphism $F : G \to Aut(G)$ using "conjugation in G"
define F by $g \longmapsto f_g$ where $f_g : G \to G$ is "conjugation by g"
$x \longmapsto f_g(x) = g x g^{-1}$

Claim $f_g \in Aut(G)$
Pf: $f_g(xy) = g x y g^{-1} = g x g^{-1} g y g^{-1} = f_g(x) f_g(y)$. This is a bijection whose inverse is $f_{g^{-1}}$ so $f_g \in Aut(G)$.

defn $F : G \to Aut(G)$ by $g \mapsto f_g$. This is a group homomorphism. Must check $F(gh) = f_{gh} \overset{?}{=} F(g) F(h) = f_g \circ f_h$. For any $x \in G$ $f_{gh}(x) = (gh) x (gh)^{-1} = g h x h^{-1} g^{-1}$
$= g (h x h^{-1}) g^{-1} = g (f_h(x)) g^{-1} = f_g \circ f_h(x)$.

ex of the simplest homomorphism $f : G \to G'$ s.t. $f(g) = e'$ $\forall g \in G$.

Note if G is abelian $F(g) = id \in Aut(G)$ for all $g \in G$ so this map is trivial.

ex. $G = S_n$ $n \geq 3$ is non-abelian. Strangely $f : G \to Aut(G)$ is an isomorphism except when $n = 6$ when it is not onto.

ex of homomorphisms ① $GL_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^* = GL_1(\mathbb{R})$
$A \longmapsto \det(A)$
$\det(AB) = \det A \cdot \det B$

note for any $a \in \mathbb{R}^*$ $\det\begin{pmatrix} a & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} = a$ so map is onto. Not injective though for $n > 1$.

② $S_n \to GL_n(\mathbb{R})$
$p \longmapsto A_p =$ permutation matrix of p (see Artin ch 1 §4)

$A_p = (a_{ij})$ where $a_{ij} = \begin{cases} 1 & \text{if } p(j) = i \\ 0 & \text{else} \end{cases}$ note: must check that A_p are invertible!

Compose ① and ② $S_n \xrightarrow[\text{perm}]{} GL_n(\mathbb{R}) \xrightarrow[\det]{} \mathbb{R}^\times$
$p \longmapsto \det(A_p) = \pm 1$ (see ch 1 §4)
[note $\det(A_p) \neq 0 \Rightarrow A_p \in GL_n(\mathbb{R})$]

This composition is the sign map

③ $S_n \xrightarrow{\text{sign}} \{\pm 1\} \subset \mathbb{R}^*$ and is onto for $n \geq 2$
$\det\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = -1$

Given a homomorphism $f: G \to G'$ define 2 subgroups

The kernel of $f \subset G$ is the set $\{g \in G : f(g) = e'\}$. [Note f an isom $\Rightarrow \ker f = \{e\}$]

The image of $f \subset G'$ is the set $\{g' \in G' : \exists\, g \in G \text{ s.t. } f(g) = g'\}$

We'll see that the kernel of f is much more interesting. Any map has an image but only groups have a special identity element and hence a kernel.

Note $\operatorname{im} f = \{e'\} \Rightarrow f =$ trivial homomorphism ; $\operatorname{im} f = G' \Rightarrow f$ is onto

ex ① $\ker(\det) = SL_n(\mathbb{R}) = \{A : \det A = 1\}$ the special linear group

② $\ker f = \{e\}$ ③ $\ker f = \{p \in S_n : \det A_p = \text{sign}(p) = +1\} = A_n$ the alternating group

e.g. $A_3 \subset S_3$ is the subgroup $\{e, (123), (132)\}$

In general $A_n \subset S_n$ has order $\frac{n!}{2}$. We'll see this later. Note if $p \in A_n$ we say it is even and if $p \notin A_n$ we say it is odd. Note product of two evens is even and product of two odds is even and product of an even and an odd is odd, just like with addition of integers.

Why kernels are interesting.

Let $H \subset G$ a subgroup. Write $f_g(H) = gHg^{-1} = \{ghg^{-1} : h \in H\}$ another subgroup of G

defn H is normal in G, $H \triangleleft G$ if $gHg^{-1} = H$ for all $g \in G$

Prop The kernel of f is always a normal subgroup.

Pf next time