Math 122 Friday, September 30

$H < G = \bigcup_{G/H} aH$

G/H — bH; $aH = a'H$; $H = eH = hH$

Note: There is no natural coset representative i.e., there is no canonical choice for a in aH. So G/H should not be regarded as a subset of G in any way.

Can we define a group structure on the set G/H of distinct cosets?

First guess for a product: $(aH)\cdot(bH) \overset{?}{=} (ab)H$
Looks nice but what if we had chosen a' instead of a and b' instead of b?

For this to be well defined need $a'b' \in abH$ for any $a' \in aH$, $b' \in bH$.
I.e., $aH \cdot bH$ = subset of G consisting of all products $\{ahbh' : h, h' \in H\} \overset{?}{=}$ the single coset abH

False when H is not normal in G. In this case $\exists\, a \in G : aHa^{-1} \neq H$. Consider the product $(aH)(a^{-1}H) = (aHa^{-1})H$. Because $aHa^{-1} \neq H$, it contains some $b \notin H$. Hence $(aHa^{-1})H$ cannot be a single coset because, on the one hand it contains bH and on the other hand, it contains $(aea^{-1})H = H$ and $bH \neq H$ as $b \notin H$.

But when $H \triangleleft G$, this product is a single coset. If $H \triangleleft G$ then $aH = Ha$, so $(aH)(bH) = (Ha)(bH) \overset{!}{=} (Hab)H = (abH)H = abH$.

Now that we have a well defined product we need to check that it is
1) associative $(aH)(bH\,cH) = (aH)(bcH) = a(bc)H = (ab)cH = (abH)(cH) = (aH\,bH)cH$
2) identity $eH\,aH = aH$
3) inverses $(aH)^{-1} = a^{-1}H$

When $H \triangleleft G$ we have defined a group structure on the set G/H of distinct cosets for H called a quotient group.

Prop There is a surjective group homomorphism $f : G \to G/H$ with $\ker f = H$.

Corollary Any normal subgroup of G is the kernel of a group homomorphism.

Pf of Prop) Define $f(a) = aH$. This is clearly surjective. Note $f(ab) = abH = aH\,bH = f(a)f(b)$ so this is a homomorphism. Also $\ker f = \{a : f(a) = e' \text{ (in } G/H) = H\} = \{a \in H\} = H$.

Note again: G/H is not a subgroup of G!

Diversion on sub-quotients of G = quotient of a subgroup

$G > K > H$ K a subgroup, H normal in G. So can define K/H (a subquotient)

Ex homology of abelian groups (a complex) $A \xrightarrow{f} B \xrightarrow{g} C$ Suppose $g \circ f = 0$.
Then $\operatorname{im} f \subset \ker g \subset B$
The subquotient $H = \ker g / \operatorname{im} f$ = homology of the complex

While we're on the subject of subquotients, let's note something actually relevant to this course.

Let $H \triangleleft G$ and $H \subset K \subset G$ as before (so $H \triangleleft K$).
Note if $a, b \in K$ then $ab \in K$ so $aH, bH \in K/H \Rightarrow (aH)(bH) \in K/H$.
In particular $K/H \subset G/H$ is a subgroup.

$\exists$ a bijection $\{$subgroups $K \subset G$ containing $H\} \longleftrightarrow \{$subgroups of $G/H\}$ that is order preserving: $K \subset K' \subset G \longrightarrow K/H \subset K'/H \subset G/H$.

ex of quotient group $G = \mathbb{Z}$ with addition. Recall all $H \subset G$ have the form $n\mathbb{Z}$.

$G/H = \mathbb{Z}/n\mathbb{Z}$ = finite group of order $n = \{n\mathbb{Z}, n\mathbb{Z}+1, n\mathbb{Z}+2, \dots, n\mathbb{Z}+(n-1)\}$
where $n\mathbb{Z}+a$ = all integers leaving a remainder of a when divided by n.

Often write $\mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$ using as coset representatives the smallest positive integers in each coset.

Note $\mathbb{Z}/n\mathbb{Z}$ cyclic generated by $1+n\mathbb{Z}$. In particular, there is a finite cyclic group of any order.

What are the subgroups of $\mathbb{Z}/n\mathbb{Z}$? Subgroups of $\mathbb{Z}$ have form $K = d\mathbb{Z}$ $d \geq 1$. When does $K > H$? True iff generator n of $H = n\mathbb{Z}$ is in K, i.e., if $n \in d\mathbb{Z}$, i.e., when d divides n. So $\mathbb{Z}/n\mathbb{Z} > d\mathbb{Z}/n\mathbb{Z}$ for all $d \geq 1$ which divide n. $d=1 \Rightarrow \mathbb{Z}/n\mathbb{Z}$, $d=n \Rightarrow n\mathbb{Z}/n\mathbb{Z} = \{e\}$

Generally $H \triangleleft G$ gives groups H and G/H but these do not determine G even though $\#G = \#H \cdot \#(G/H)$

ex $G = S_3$, $G' = \mathbb{Z}/6\mathbb{Z}$. Clearly $G \neq G'$ because G is not abelian.

$S_3 \triangleright A_3 = \{e, (123), (132)\}$ cyclic of order 3 $\quad G/H = \langle \pm 1 \rangle$

$\mathbb{Z}/6\mathbb{Z} \triangleright 2\mathbb{Z}/6\mathbb{Z} = \{0, 2, 4\} \subset \{0, 1, 2, 3, 4, 5\}$ cyclic of order 3 $\quad G'/H' = \mathbb{Z}/2\mathbb{Z}$
(any subgroup of an abelian group is normal)

defn A group G is <u>simple</u> if the only normal $H \triangleleft G$ are G and $\{e\}$.

ex! Simple groups $\mathbb{Z}/p\mathbb{Z}$, p a prime; A_n $n \geq 5$ and 20 more infinite chains; 26 "sporadic" groups including the Monster group

non-ex! S_n, $GL_n(\mathbb{R})$, and $GL_n(\mathbb{Z}/p\mathbb{Z})$