123 Solution Set 2

7.4.4 Because X^*AX is real, we have $X^*AX = (X^*AX)^* = X^*A^*X$, which implies $X^*(A - A^*)X = 0$ for all X. Thus it suffices to show that $X^*BX = 0$ for all X implies that $B = 0$. We have

$$\begin{aligned} (X^* + Y^*)B(X + Y) &= 0 \\ \Rightarrow X^*BX + X^*BY + Y^*BX + Y^*BY &= 0 \\ \Rightarrow X^*BY + Y^*BX &= 0. \end{aligned}$$

Thus, substituting $e_j = X$, $e_k = Y$, we have $b_{jk} + b_{kj} = 0$. Substituting $e_j = X$, $ie_k = Y$, we have $ib_{jk} - ib_{kj} = 0$. These two equations together imply that B is identically zero.

7.4.6 Write $v_2' = c_1v_1 + c_2v_2$ for some $c_1, c_2 \in \mathbb{C}$. We must have $0 =< v_1, v_2' >= |c_1| < v_1, v_1 > +|c_2| < v_1, v_2 >$, so $c_1 = 0$, and $|v_2'| = 1$, which implies $|c_2| = 1$.

7.4.11 If A is positive definite hermitian, then it corresponds the standard hermitian inner product under a change of basis; that is, $Q^*AQ = I$ for some invertible Q, so just let $P = (Q)^{-1}$. Conversely, if $A = P^*P$, then $(P^*)^{-1}AP^{-1} = (P^{-1})^*AP^{-1} = I$, so A is just the matrix of the standard hermitian inner product with respect to another basis; that is, it is positive definite hermitian.

7.4.15 Note first that $< x, y >= \overline{< y, x >}$ implies that $\{x, y\} = \{y, x\}$. The axioms for bilinearity are straightforward since $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$. For positive definiteness, note that $< x, x >= \{x, x\}$. The imaginary part is a skew-symmetric bilinear form.

7.4.17.a It's clear that we have conjugate linearity in the first variable and linearity in the second variable by the definition of * and the fact that the trace is additive. For hermitian symmetry, we have

$$\overline{\langle A, B\rangle} = \overline{\text{trace}(A^*B)} = \overline{\text{trace}(\overline{A}^tB)} = \text{trace}(A^t\bar{B}) = \text{trace}((A^t\bar{B})^t) = \text{trace}(\bar{B}^tA) = \langle B, A\rangle.$$

Now note that $\langle A, A\rangle = \text{trace}(A^*A)$, which is just the sum of the standard hermitian norms of the columns of A. It follows that $\langle A, A\rangle \geq 0$, with equality if and only if $A = 0$. Thus the form is positive definite, and the signature is just $(n^2, 0)$ because the dimension of $\mathbb{C}^{n\times n}$ is n^2.

7.4.17.b Again, this is a hermitian form. Conjugate linearity in the first variable and linearity in the second follow by the definition of conjugation and linearity of the trace. Hermitian symmetry is even easier than before:

$$\overline{\langle A, B\rangle} = \overline{\text{trace}(\bar{A}B)} = \text{trace}(A\bar{B}) = \text{trace}(\bar{B}A) = \langle B, A\rangle.$$

Here we used the fact that $\text{trace}(AB) = \text{trace}(BA)$ (if you don't see this, just explicitly write out both sides of the equation in terms of matrix entries).

We now determine the signature. Let E_{ij} denote, as usual, a matrix with a 1 in the ij entry and zeros elsewhere. We note that if we define the sets of matrices $P := \{E_{ij} + E_{ji}\}_{1\leq i<j\leq n}$, $D := \{E_{jj}\}_{1\leq j\leq n}$, $N := \{E_{ij} - E_{ji}\}_{1\leq i<j\leq n}$, then $P \amalg D \amalg N$ is a basis for the space of $n \times n$ complex matrices. Note also that $P \amalg D \amalg N$ forms

an orthogonal basis for this space of matrices, and further, for all $a \in P, b \in D$ we have $\langle a, a\rangle = \langle b, b\rangle = 1$ and for all $c \in N$ we have $\langle c, c\rangle = -1$. By counting these sets, we see that the signature is $(\frac{n^2+n}{2}, \frac{n^2-n}{2})$.

7.4.20 Evidently A^*A is hermitian, so by the spectral theorem there exists a unitary P such that P^*A^*AP is diagonal. We have $P^*(I + A^*A)P = I + P^*A^*AP$. The diagonal matrix P^*A^*AP has as jth diagonal entry the norm of the jth column with respect to the standard hermitian inner product; thus, in particular, all of its entries are positive, which immediately implies that $I + +P^*A^*AP$ (and hence also $I + A^*A$) are both nonsingular.

7.5.10 We take $\perp$ to be with respect to the standard hermitian product $<,>$. Suppose $x \in \ker A$. Then $Ax = 0 \Rightarrow < Ax, y >= 0$ for all y which implies $< x, A^*y >= 0$ for all y, which implies $x \in (ImA^*)^\perp$.

Conversely, suppose $x \in (ImA^*)^\perp$. Then, for all y, $< x, A^*y >= 0 \Rightarrow < Ax, y >= 0$, which implies, since $<,>$ is nondegenerate, that $Ax = 0$.

7.5.11 We have to check that $(c_{rs})A^*A = I$. Note that $c_{rs} = \sum_{k=1}^n \overline{a_{kr}} a_{ks} = \frac{1}{n}\sum_{k=1}^n e^{-2\pi ikr/n} e^{2\pi iks/n}$. First we note that if $r = s$, then each of these terms is just 1, so we obtain $c_{rr} = \frac{n}{n}$. If $s \neq r$, then this is just a sum $\frac{1}{n}\sum_{k=1}^n (e^{2\pi ir-s/n})^k$. Because $r \neq s$, $0 \leq r, s \leq n$, this is just a sum

$$\frac{1}{n}(1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1})$$

where ζ is some nth root of unity other than 1. Such a sum is well-known to be zero; one can see this by noting $x^n - 1 = (x - 1)(1 + x + \cdots + x^{n-1})$, for example.

7.5.14 First choose $Q \in U(n)$ such that Q^*BQ is diagonal using the spectral theorem. For each eigenvalue λ_j of B, let V_{λ_j} be the associated eigenspace. We may assume, perhaps by changing Q by a permutation matrix, that

$$Q^*BQ = \begin{pmatrix} \begin{bmatrix} \lambda_1 & & \\ & \cdots & \\ & & \lambda_1 \end{bmatrix} & & & \\ & \begin{bmatrix} \lambda_2 & & \\ & \cdots & \\ & & \lambda_2 \end{bmatrix} & & \\ & & \cdots & \\ & & & \begin{bmatrix} \lambda_k & & \\ & \cdots & \\ & & \lambda_k \end{bmatrix} \end{pmatrix}$$

This is just the matrix of B with respect to the basis $V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_k}$.

For all $v \in V_{\lambda_j}$, we have $\lambda_j Av = A\lambda_j v = ABv = BAv$, that is, Av is a eigenvector of B with eigenvalue λ_j, and hence A preserves V_{λ_j}. Thus $A|_{V_{\lambda_j}}$ is a hermitian linear operator on V_{λ_j}, and is diagonalizable by the spectral theorem, say $P_{\lambda_j}^* A|_{V_{\lambda_j}} P_{\lambda_j}$ is diagonal for $P_{\lambda_j} \in U(\dim(V_{\lambda_j}))$. Thus, if we define

$$P := \begin{pmatrix} P_{\lambda_1} & & & \\ & P_{\lambda_2} & & \\ & & \dots & \\ & & & P_{\lambda_k} \end{pmatrix}$$

it follows that $(QP)^*AQP$ is diagonal, and, further, $(QP)^*BQP = Q^*BQ$, because P preserves the eigenspaces of B.

7.6.1

We may write this quadric as $X^TAX + BX + c$, where

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

and $B = (3, 0, 1)$. By the Spectral theorem, there exists an orthogonal matrix P such that PAP^T is diagonal. However, we don't have to diagonalize with respect to an orthogonal to determine the type of the quadric (see p. 258 of Artin). Consider the nonorthogonal matrix

$$P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 5 & 1 \end{pmatrix}$$

We will use this matrix to transform the quadric.

$$P^TAP = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

Note $(3, 0, 1)P = (3, -6, 1)$.

Our original quadric was $X^TAX+BX-6 = 0$. With the change of basis $X = PX'$ we have the new, equivalent quadric

$$X'^T(P^TAP)X + (BP)X + c = 0 = 5x'^2 + 20y'^2 - 4z'^2 + 3x' + 2y' - 2z' - 6 = 0$$

We complete the square:

$$5(x' + \frac{3}{20})^2 - \frac{9}{20} + 20(y' + \frac{1}{20}^2 - \frac{1}{20} - 4(z' + \frac{1}{4})^2 + \frac{1}{4} - 6 =$$

$$5(x' + \frac{3}{10})^2 + 20(y' + \frac{1}{20})^2 - 4(z' + \frac{1}{4})^2 - \frac{25}{4} = 0$$

Thus we have a one-sheeted hyperboloid. Notice that in completing the squares, we affected the constant coefficient. Therefore, it is not sufficient to compute the signature of the original form to distinguish between the one-sheeted and two-sheeted hyperboloids (I don't think I bothered with this point in grading the homeworks, but you should be aware of it).

7.9.5.a

The problem is that X^tX is not necessarily a unit (ie nonzero). For example, $(i,1)^t(i,1) = 0$.

7.9.5.b

Replace ^{t} by * everywhere in the argument, and let A be a unitary matrix (which means that A could be real orthogonal). Then the same proof shows that $X^*P^*X = (PX)^*X = \overline{\lambda}X^*X$ and $X^*P^*X = X^*P^{-1}X = \lambda^{-1}X^*X$. $X^*X > 0$ in this case, so we have $\overline{\lambda} = \lambda^{-1}$ which implies $|\lambda| = 1$.