math123, Abstract Algebra II

Exam 2

Your name:

Problem 1 (15pt)

Let H be a normal subgroup of index 2 of a finite group G, and let $\rho: G \to GL(V)$ be a finite–dimensional representation of G. Define ρ' by:

$$\begin{aligned} \rho'(h) &= \rho(h) , \quad \text{for } h \in H , \\ \rho'(g) &= -\rho(g) , \quad \text{for } h \notin H . \end{aligned}$$

(1) Show that ρ' is a representation of G.
(2) Show that ρ' is irreducible if and only if ρ is irreducible.

Solutions

(1) Clearly $\rho'(1) = \rho(1) = \mathbb{1}$. We need to show $\rho'(g_1 g_2) = \rho'(g_1)\rho'(g_2)$. There are four cases to consider:
  (a) $g_1, g_2 \in H$, which implies $g_1 g_2 \in H$ (since H is submodule). In this case:
  $$\rho'(g_1 g_2) = \rho(g_1 g_2) = \rho(g_1)\rho(g_2) = \rho'(g_1)\rho'(g_2) .$$
  (b) $g_1 \in H$, $g_2 \notin H$, which implies $g_1 g_2 \notin H$ (since the right coset Hg_2 is invariant under left multiplication by elements of H). In this case:
  $$\rho'(g_1 g_2) = -\rho(g_1 g_2) = \rho(g_1)(-\rho(g_2)) = \rho'(g_1)\rho'(g_2) .$$
  (c) $g_1 \notin H$, $g_2 \in H$, which implies $g_1 g_2 \notin H$ (since the left coset $g_1 H$ is invariant under right multiplication by elements of H). In this case:
  $$\rho'(g_1 g_2) = -\rho(g_1 g_2) = (-\rho(g_1))\rho(g_2) = \rho'(g_1)\rho'(g_2) .$$
  (d) $g_1, g_2 \notin H$, which implies $g_1 g_2 \in H$ (here we need to use that H has index two. In particular $g_2 \in g_1^{-1} H$). In this case:
  $$\rho'(g_1 g_2) = \rho(g_1 g_2) = (-\rho(g_1))(-\rho(g_2)) = \rho'(g_1)\rho'(g_2) .$$

(2) **First way:** It suffices to notice that the square norm of the charaters of ρ and ρ' coincide: $< \chi, \chi >=< \chi', \chi' >$, and a representation of character χ (of a finite group) is irreducible if and only if $< \chi, \chi >= 1$.
Second way: It suffices to notice that a subspace $U \subset V$ is invariant under the action of $\rho(g)$ is and only if it is invariant under the action of $\rho'(g)$ (and this is obvious, by definition of ρ').

Problem 2 (20pt)

Let ρ be an irreducible representation of a finite group G on a vector space V, and let T be the linear operator on V defined by

$$T \;=\; \sum_{g\in G} \rho_g \ .$$

(1) Prove that T is an invariant operator on V.
(2) What does Schur's Lemma tell us about T?
(3) Using the orthogonality relations, show that if ρ is the non–trivial representation, then the trace of T is zero.
(4) Prove that, is ρ is not the trivial representation, then $T = 0$.

Solution

(1)

$$hTh^{-1} \;=\; \sum_{g\in G} h\rho_g h^{-1} \;=\; \sum_{g\in G} \rho_{hgh^{-1}} \;=\; \sum_{k\in G} \rho_k = T$$

(2) Since V is irreducible, we have

$$T \;=\; \lambda \mathbb{1} \ , \quad \lambda = \frac{\mathrm{Tr} T}{\dim V}$$

(3)

$$\mathrm{Tr} T \;=\; \mathrm{Tr} \sum_{g\in G} \rho(g) \;=\; \sum_{g\in G} \mathrm{Tr}\rho(g) \;=\; \sum_{g\in G} \chi(g) \;=\; |G| < 1, \chi > \;=\; 0$$

(4) It follows by (2) and (3)

Problem 3 (20pt)

(1) Determine the character table for the Klein Four Group $K = C_2 \times C_2$.
(2) Consider a group G of class equation $8 = 1 + 1 + 2 + 2 + 2$. Determine its character table. (**Hint.** What is the center of the group? And the quotient by the center?)
(3) Let V be a representation of G with character:

	(1)	(1)	(2)	(2)	(2)
χ	8	0	4	2	2

Write the decomposition of V as direct sum of irreducible representations of G.

Solution

(1) Since K is abelian of order 4, there are exactly 4 irreducible representations, all of dimension 1. Since every element (not equal to 1) has order 2, all values of characters are $\chi(g) = \pm 1$. In other words, the character table will look like:

	1	x	y	xy
χ_1	1	1	1	1
χ_2	1	± 1	± 1	± 1
χ_3	1	± 1	± 1	± 1
χ_4	1	± 1	± 1	± 1

The only way to make all the rows of this table orthogonal, is by taking:

	1	x	y	xy
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

(2) By the dimension formula, we know that there are exactly 5 irreducible representations of G, of dimensions $1 = d_1 \le d_2 \le d_3 \le d_4 \le d_5$, satisfying the identity:

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 \;=\; 8\ .$$

The only solution to this equation is $d_1 = d_2 = d_3 = d_4 = 1,\ d_5 = 2$.

The center of the group G is $Z = \{1, a\}$, and the quotient group if $G/Z \simeq K$. This means that the first four irreducible representations of G are obtained by taking the irreducible representations of K, and letting Z

act trivially. In other words the character table will look like:

	(1)	(1)	(2)	(2)	(2)
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	*	*	*	*

The only way to make the last row orthogonal to the previous ones is by taking:

	(1)	(1)	(2)	(2)	(2)
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

(3) Since we have $\chi = 3\chi_1 + \chi_2 + 2\chi_5$, we immediately conclude:

$$V = V_1^{\oplus 3} \oplus V_2 \oplus V_5^{\oplus 2} .$$

Problem 4 (15pt)

Let R be a Euclidean domain, and let M be a finitely generated module over R. For each of the following statements, determine whether it is necessarily true (by proving it) or it can be false (by finding a counterexample).

(1) M has a basis.
(2) M is a submodule of a free module R^n.
(3) M is a quotient module of a free module R^n.
(4) If M is free of rank n, and we have a submodule $R^k \subset M$, then M/R^k is free of rank $n-k$.
(5) If M is free of rank n, and we have a bijective module homomorphism $R^k \to M$, then $k = n$.

Solution

(1) False. **Ex:** $\mathbb{Z}/2\mathbb{Z}$.
(2) False. **Ex:** $\mathbb{Z}/2\mathbb{Z}$.
(3) True. **Proof:** there is a surgective map $R^n \to M$, given by $e_i \mapsto v_i$.
(4) False. **Ex:** $\mathbb{Z} \simeq 2\mathbb{Z} \subset \mathbb{Z}$, but $\mathbb{Z}/2\mathbb{Z} \neq 0$.
(5) True. **Proof:** we proved that the rank of a free module is uniquely defined.

Problem 5 (15pt)

Let $L \subset \mathbb{Z}^2$ be the sublattice spanned by the vectors

$$v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad , \quad v_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} .$$

Find bases $\mathcal{B} = [e_1, e_2]$ of $\mathbb{Z}^2$ and $\mathcal{B}' = [v_1, v_2]$ of L, so that $v_1 = d_1 e_1$ and $v_2 = d_2 e_2$, with $d_1, d_2 \in \mathbb{Z}_+$.

Solution

The lattice L is spanned by the columns of the matrix:

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$$

The diagonalization algorithm gives:

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} \xrightarrow{\times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 1 \\ 5 & -1 \end{bmatrix} \xrightarrow{\times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix}$$
$$\xrightarrow{\times \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \times} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} .$$

This gives the equation

$$\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$$

Therefore, a basis of $\mathbb{Z}^2$ is given by the columns of P^{-1}:

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = [e_1 \ e_2]$$

and the corresponding basis of the lattice L is:

$$\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = [w_1 \ w_2]$$

Problem 6 (15pt)

Let V be the abelian group generated by three elements x, y, z, with relations

$$\begin{aligned} 6x + 4y + 4z &= 0 \,, \\ 2x + 2y + 8z &= 0 \,. \end{aligned}$$

Identify V as a direct sum of cyclic groups.

Solution

A presentation of the module V (over $\mathbb{Z}$) is given by the matrix:

$$A = \begin{bmatrix} 2 & 6 \\ 2 & 4 \\ 8 & 4 \end{bmatrix}$$

The diagonalization algorithm gives:

$$A \simeq \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

which implies that the module V decomposes as:

$$V \simeq \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z} \,.$$