Mathematics 113 Fall 2002

HOMEWORK ASSIGNMENT # 5
DUE, Thursday, October 24

Collaboration: On the homework sets, collaboration is not only allowed but encouraged. However, you must write up and understand your own individual homework solutions, and you may not share written solutions. If you learn how to solve a problem by talking to a classmate, CA, or looking it up in a book, you should cite the source in your homework write-up, just as you would reference your sources in a literature or history class.
Show all your working, and write up your solutions as neatly as possible.

1. (a) Let $g(z)$ be a holomorphic function in a neighbourhood of $z = a$. Suppose that $g(a) = 0$. Prove that $g(z)/(z-a)$ extends to a holomorphic function at $z = a$. (Hint: use Riemann's Removable Singularity Theorem.)

   (b) Let $f(z)$ be a holomorphic function in a neighbourhood of $z = a$ except for a singularity at $z = a$. Suppose that the limit
   $$\lim_{z \to a} (z-a)^n f(z)$$
   exists for some integer n. Using part (a), show there exists an integer $n' \leq n$ such that
   $$\lim_{z \to a} (z-a)^{n'} f(z)$$
   exists and is non-zero.

   (c) Let $f(z)$ be a holomorphic function in a neighbourhood of $z = a$ except for a singularity at $z = a$. Show that either $f(z)$ has a pole of order n at a for some integer n, or
   $$\lim_{z \to a} (z-a)^n f(z)$$
   does not exist for any n.

2. (a) If $f(z)$ is holomorphic inside and on the simple closed curve C containing $z = a$, prove that
   $$f(a)^n = \frac{1}{2\pi i} \oint_C \frac{f(z)^n}{(z-a)} dz$$

   (b) Use (a) to prove that $|f(a)|^n \leq LM^n/(2\pi D)$, where D is the minimum distance from a to the curve C, L is the length of C and M is the maximum value of $|f(z)|$ on C.

   (c) Use (b) to prove that $|f(a)| \leq M$. In other words, the maximum value of $|f(z)|$ is obtained on its boundary. This result is known as the Maximum Modulus Principle.

(d) The maximal value of $f(z) = 1/z$ on the unit circle is 1, yet $|f(\frac{1}{2})| = 2$. Explain why this does not contradict (c).

(e) (**Fundamental Theorem of Algebra**) Using the maximum modulus principle, prove the Fundamental Theorem of Algebra. (Hint: Consider $f(z) = 1/P(z)$.)

(f) Let $f(z)$ be holomorphic on and inside C. Let M be the maximal value of $|f(z)|$ on C. Suppose that $|f(a)| = M$ for some a *inside* C. Prove that $f(z)$ is constant.

3. Let

$$f(z) = \frac{z}{e^z - 1} + z/2 = 1 + \frac{z^2}{12} - \frac{z^4}{720} + \frac{z^6}{30240} + \ldots = \sum_{n=0}^{\infty} \frac{z^n B_n}{n!}$$

(a) Prove that $f(-z) = f(z)$.

(b) Prove that $B_n = 0$, if n is odd.

(c) Write $\tan z$ in terms of e^{iz}, and use this to find the Taylor series of $\tan z$ around $z = 0$ in terms of the Bernoulli numbers B_n.

(d) What is the radius of convergence of $\tan z$ around $z = 0$?