MOCK FINAL EXAM

Three Hours

Requirements: Calculators are not allowed. Show all your working, and submit partial solutions. Write as neatly as possible. Partial marks will be awarded, when appropriate. If you use a theorem, make sure to state it. All questions are worth the same amount. Do not feel you need to write up the questions in order.

1. Let $F(z)$ be holomorphic function. Suppose that F satisfies the identity
$$F(z) = F(e^z - 1).$$
Prove that F is constant.

2. Show that the functions $(\sin\sqrt{z}/\sqrt{z})$ and $\sqrt{\sin z/z}$ are holomorphic in a neighbourhood of zero and calculate their radius of convergence.

3. Let $P(x, y)$ be a polynomial in two variables with real coefficients. Prove there exists a holomorphic function $f(x + iy) = u(x, y) + iv(x, y)$ with $u(x, y) = P(x, y)$ if and only if
$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0.$$

4. Let n be a non-negative integer. Compute the integral
$$\int_0^\infty \frac{1}{(1+x^2)^{n+1}}\ dx.$$

5. Prove that
$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\ dx = \frac{-\pi}{4}.$$

6. Evaluate the integral
$$\int_0^\infty \sin x^3\ dx.$$

7. Let f be a holomorphic function such that $|f(z+1)| \leq |f(z)|$ for all $z \in \mathbb{C}$. Suppose moreover that $f(0) = f(1) = 1$. Prove that $f(z+1) = f(z)$.