Problem Set 6 Solution Set

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Math 113: Complex Analysis, Fall 2002

1. Let $f(z)$ be a holomorphic function on the unit disc such that $|f(z)| = 1$ for all $|z| \leq 1$. Prove that $f(z)$ is constant, using

   (a) The Open Mapping Theorem

   Solution. The set $|z| < 1$ is open and is mapped to an open set of the complex plane since f is holomorphic and non-constant. Furthermore, the image of this set is connected since f is continuous. But no connected subset of the unit circle is open in the complex plane. Any neighborhood of a point on the circle contains points outside the circle. Therefore f is constant. □

   (b) The Cauchy–Riemann equations, and the identity

   $$|f(x+iy)| = u(x,y)^2 + v(x,y)^2.$$

   Solution. In the unit disc we have $1 = u^2 + v^2$. Real variable differentiation yields the two equations

   $$u\,u_x + v\,v_x = 0,$$
   $$u\,u_y + v\,v_y = 0.$$

   Assume that f is not constant. Now multiply the first equation by u and substitute $v_x = -u_y$ (one of the two CR equations) to get

   $$u^2\,u_x - uv\,u_y = 0.$$

   Similarly, multiply the second equation above by v and substitute $v_y = u_x$ to obtain

   $$uv\,u_y + v^2\,u_x = 0.$$

   Adding the last two equalities we get

   $$u_x(u^2 + v^2) = u_x = v_y = 0.$$

   Our original two equations become $v\,v_x = u\,u_y = 0$ so $v^2\,v_x - u^2\,u_y = v_x(v^2 + u^2) = v_x = -u_y = 0$. Hence both u and v are constant on the unit disc. So f is constant on an infinite number of points and therefore constant in all of $\mathbb{C}$. □

2. (a) Let n be a positive integer. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{1}{x^{2n}+1}\,dx.$$

Solution. Consider the integral

$$\oint_C \frac{1}{z^{2n}+1}\,dz,$$

where C is the closed contour the joins a semicircular arc Γ_R of radius R in the upper half plane centered at the origin with the real line segment $\Gamma = [-R, R]$. The Cauchy Integral Theorem tells us that

$$\int_{\Gamma_R} \frac{1}{z^{2n}+1}\,dz + \int_{\Gamma} \frac{1}{z^{2n}+1}\,dz = 2\pi i \sum \text{Res of } \left(\frac{1}{z^{2n}+1}\right) \text{ inside } C.$$

Since $n \geq 1$, the polynomial $(z^{2n}+1) - 1$ always has degree at least 2, we showed in class that in such a case

$$\lim_{R\to\infty} \int_{\Gamma_R} \frac{1}{z^{2n}+1}\,dz = 0$$

We then conclude that

$$\int_{-\infty}^{\infty} \frac{1}{x^{2n}+1}\,dx = \lim_{R\to\infty} \int_{\Gamma} \frac{1}{z^{2n}+1}\,dz = \sum \text{Res of } \left(\frac{1}{z^{2n}+1}\right) \text{ inside } C.$$

It remains to calculate the sum of the residues above. The function we are integrating has simple poles at $e^{\pi i(2k+1)/2n}$, $k = 0, \ldots, 2n-1$. Only the first n of these poles are inside the region bounded by C. The residue at any one of these poles is

$$\begin{aligned}
\lim_{z\to e^{\pi i(2k+1)/2n}} \frac{(z - e^{\pi i(2k+1)/2n})}{z^{2n}+1} &= \lim_{z\to e^{\pi i(2k+1)/2n}} \frac{1}{2nz^{2n-1}} \\
&= \frac{1}{2ne^{\pi i(2k+1)(2n-1)/2n}} = -\frac{1}{2ne^{-\pi i(2k+1)/2n}}.
\end{aligned}$$

Hence the sum of the residues of $1/(z^{2n}+1)$ inside C is

$$\begin{aligned}
\sum_{k=0}^{n-1} -\frac{1}{2ne^{-\pi i(2k+1)/2n}} &= -\frac{1}{2n}\sum_{k=0}^{n-1} e^{\pi i(2k+1)/2n} \\
&= -\frac{1}{2n}\,\frac{e^{\pi i/2n}(1-e^{\pi i})}{1-e^{\pi i/n}} \\
&= -\frac{e^{\pi i/2n}}{n(1-e^{\pi i/n})} \\
&= -\frac{1}{n(e^{-\pi i/2n} - e^{\pi i/2n})} \\
&= -\frac{i}{2n\sin\pi/2n}.
\end{aligned}$$

Finally,

$$\int_{-\infty}^{\infty} \frac{1}{x^{2n}+1}\,dx = 2\pi i \cdot \frac{-i}{2n\sin\pi/2n} = \frac{\pi}{n\sin(\pi/2n)}$$

□

(b) Given that

$$\lim_{n\to\infty} \frac{1}{x^{2n}+1} = \begin{cases} 0 & |x| > 1, \\ 1 & |x| < 1, \end{cases}$$

find the limit of the integral from part (a) as n approaches infinity, and show that it agrees with your answer.

Solution. If this were a course in real analysis, it would take a while to justify why we can exchange the following limit and integral:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{1}{x^{2n}+1}\, dx = \int_{-\infty}^{\infty} \lim_{n\to\infty} \frac{1}{x^{2n}+1}\, dx.$$

We would have to discuss how uniform convergence in this case still makes the exchange possible even though the integral is improper. Those of you with strong souls (and stomachs) could even consult the beloved *Hubbard & Hubbard*. Fortunately, this is a course in complex analysis. Given the above exchange we see that the limit of the integral is 2, with the given information.

Now we compute the same limit using our answer from part (a).

$$\begin{aligned} \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{1}{x^{2n}+1}\, dx &= \lim_{n\to\infty} \frac{\pi}{n \sin(\pi/2n)} \\ &= \lim_{y\to 0} 2\frac{y}{\sin y} = 2. \end{aligned}$$

□

3. Let $f(z)$ be a holomorphic function such that $f(z) \neq 0$ for all z. Let $g(z) = \log f(z)$. Prove that $g(z)$ extends to a holomorphic function for all z.

Solution. Since $f(z) \neq 0$ for any z, the quotient $f'(z)/f(z)$ is holomorphic. Furthermore, the function given by the line integral

$$F(z) = \int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta$$

is holomorphic. A quick way to see this, for example, is to recall a holomorphic function has a uniformly convergent power series expansion on the domain where the function is holomorphic (we showed this in section a while back). We can thus integrate the series term by term and obtain a new uniformly convergent power series which is again a holomorphic function. We also proved this in class (in a different way), so I won't go through the details.

Choose a branch cut to make the logarithm a holomorphic function, e.g., $B = (-\infty, 0]$. Let z_0 be a point such that $f(z_0) \notin B$. Note that such a point exists, otherwise f maps the complex plane to a closed subset of itself, in contradiction with the Open Mapping Theorem.

Now observe that

$$(\log f(z))' = F'(z) = \frac{f'(z)}{f(z)},$$

where the second equality is just the fundamental theorem of calculus for complex variables (see Thm 2, p.97 of Spiegel). Hence $\log f(z)$ and $F(z)$ differ by a constant on $\mathbb{C} - B$:

$$\log f(z) = F(z) + k =: G(z).$$

Now here's the key: the function $G(z)$ makes sense for all z; it is holomorphic and coincides with $\log f(z)$ on the region where the latter is defined. Thus, it is the holomorphic extension to the complex plane that we are looking for. $\square$