Homework 10

Math 124, Fall 2005

Due Wednesday, November 30th

No late assignments will be accepted.

1. Prove that $n! = m^k$ is impossible in integers $k, m, n > 1$.
2. Let k and r be integers greater than one. Show that there is a prime number whose representation in base r has exactly k digits.
3. For this problem include 1 as prime. Prove that every positive integer can be written as a sum of one or more distinct primes.
4. Consider the functions

$$\nu(d) = \begin{cases} 1 & d = 1, 30 \\ -1 & d = 2, 3, 5 \\ 0 & \text{otherwise.} \end{cases} \qquad N(y) = \sum \nu(d) \left[\frac{y}{d}\right]$$

   (a) Show that

$$\sum \frac{\nu(d)}{d} = 0$$

   (b) Show that N has period 30, that N takes only the values 0 or 1 and that

$$N(y) = 1 \qquad 1 \leq y < 6$$

   (c) Let

$$\begin{aligned} a_1 &= \frac{7}{15}\log 2 + \frac{3}{10}\log 3 + \frac{1}{6}\log 5 \\ b_1 &= \frac{6}{5}a_1. \end{aligned}$$

   Show that for any $a < a_1$ and any $b > b_1$ then there exists an x_0 so that for any x bigger than x_0

$$ax < \psi(x) < bx.$$

   (d) Show that the interval (x, cx) contains a prime for large x provided that $c > \frac{6}{5}$.

5. Suppose that

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1.$$

   Show that for any $c > 1$ there is a $x_0(c)$ such that for $x > x_0(c)$ the interval (x, cx) contains one prime.