Homework 12

Math 124, Fall 2005

Due Wednesday, Decmber 14th

No late assignments will be accepted.

1. (a) Show that if $s > 1$ then
$$\zeta(s) = s\int_1^\infty \frac{[u]}{u^{s+1}}du.$$
   (b) Show that if $s > 1$ then
$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}du. \tag{1}$$
   (c) Show that this integral is absolutely convergent for all $s > 0$.
   (d) Show that if we define $\zeta(s)$ using (1) for $0 < s < 1$ then
$$\zeta(s) = -s\int_0^\infty \frac{\{u\}}{u^{s+1}}du.$$
   Conclude that $\zeta(s) < 0$ for $0 < s < 1$.

2. (a) Show that for $x \geq 2$
$$\sum_{n\leq x} \frac{\sigma(n)}{n} = \frac{\pi^2}{6}x + O(\log x).$$
   You may use that $\zeta(2) = \pi^2/6$.
   (b) Let k be a fixed integer bigger than 1. Show that the number of kth power-free numbers $n \leq x$ is
$$\frac{x}{\zeta(k)} + O(x^{1/k}).$$

3. Let
$$D(x) = \sum_{n\leq x} d(n).$$
   (a) Show that
$$\sum_{n\leq x} \frac{d(n)}{n} = \frac{D(x)}{x} + \int_1^x \frac{D(u)}{u^2}du$$
   (b) Deduce that
$$\sum_{n\leq x} \frac{d(n)}{n} = \frac{1}{2}(\log x)^2 + O(\log x).$$

(c) Let $d_k(n)$ be define as in the last assignment. Deduce that

$$\sum_{n\leq x} d_3(n) = \frac{1}{2}x(\log x)^2 + O(x\log x).$$