# MATH 124 HOMEWORK #10

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(1) Let $\ell = \lceil n/2 \rceil$. We know that between $\ell$ and $2\ell \geq n$ there is a prime $p$. Thus in the prime factorization of $n!$ $p$ appears exactly once. Thus if we could write $n! = m^k$ we would have $k = 1$, a contradiction. Thus we cannot write $n! = m^k$ with $n, m, k > 1$.

(2) By Bertrand's postulate, we know that there is a prime number $p$ between $r^{k-1}$ and $2r^{k-1} \leq r^k$. Then $p = r^{k-1} + \ell$, where $\ell < r^k$, which means that the representation of $p$ in base $r$ has $k$ digits, as desired.

(3) We will prove this by induction. Clearly, $1, 2, 3$ can be represented in this way. Suppose that we can represent all numbers up to $n - 1$ in this way. Consider $n$. If $n$ is prime, we are done. Otherwise, let $k = \lceil n/2 \rceil$. We know that there is a prime $p$ between $k$ and $2k$. $2k = n$ or $n + 1$, so we know that $p$ will be strictly less than $n$ and strictly bigger than $n/2$. Now consider $n - p$; this is strictly less than both $n$ and $p$, which means that we can apply the induction hypothesis to it, and none of the primes in the sum will equal $p$. Therefore we can write $n$ as a sum of distict primes, and we are done.

(4) (a)

$$\sum \frac{\nu(d)}{d} = \frac{1}{1} + \frac{-1}{2} + \frac{-1}{3} + \frac{-1}{5} + \frac{1}{30} = 0.$$

(b) Notice that

$$\begin{aligned} N(x+30) &= \left\lfloor \frac{x+30}{1} \right\rfloor - \left\lfloor \frac{x+30}{2} \right\rfloor - \left\lfloor \frac{x+30}{3} \right\rfloor - \left\lfloor \frac{x+30}{5} \right\rfloor + \left\lfloor \frac{x+30}{30} \right\rfloor \\ &= N(x) + 30 - 15 - 10 - 6 + 1 = N(x) \end{aligned}$$

so the period of $N$ is at most 30. If the period of $N$ is smaller than 30 it must divide 30. Notice that $N(1) = 1$ while $N(16) = 0$, which means that the period does not divide 15. Notice that $N(21) = 0$, which means that the period does not divide 20. However, every proper divisor of 30 divides one of these two numbers, so the period is exactly 30.

Suppose that $x < 24$. Note that

$$\begin{aligned} N(x+6) &= \left\lfloor \frac{x+6}{1} \right\rfloor - \left\lfloor \frac{x+6}{2} \right\rfloor - \left\lfloor \frac{x+6}{3} \right\rfloor - \left\lfloor \frac{x+6}{5} \right\rfloor + \left\lfloor \frac{x+6}{30} \right\rfloor \\ &= \lfloor x \rfloor - \left( \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x+1}{5} \right\rfloor \right). \end{aligned}$$

Thus if $x \not\equiv 4 \pmod 5$, we have that $N(x+6) = N(x)$. If $x \equiv 4 \pmod 5$, then $N(x) = N(x) - 1$. Notice that, for any given $x$ between 1 and 6, we can only add 6 to it at most 4 times before we hit 30. In particular, we will subtract 1 exactly once from each one except 6, which we never subtract from. Thus if we have that for $1 \leq y < 6$ we have $N(y) = 1$ we are done. This is a simple computation, and turns out to be true. So we are done.

(c) We do a computation analogous to that of the proof of theorem 8.3. Notice that equation (8.7) gives the lower bound

$$\psi(x) \geq a_1 x - 5 \log ex$$

for $x \geq 30$. Thus for all $a < a_1$ we have $\psi(x) > ax$ for sufficiently large $x$.
An argument analogous to the given one with 3 replaced by 6 gives us the desired upper bound.

(d) Choose $a < a_1$ such that $ca > b_1$ (this is possible since $c > 6/5$), and let $b = (ca + b_1)/2$. Let

$$\epsilon = \frac{1}{2}\left(\frac{ac - b}{ac + b}\right).$$

From these definitions, we know that

$$(1 - \epsilon)ac - (1 + \epsilon)b > 0$$

which means that

$$(1 - \epsilon)\psi(cx) - (1 + \epsilon)\psi(x) > 0.$$

It remains to show that this means that $\pi(cx) - \pi(x) > 0$.
Notice that we know that

$$\lim_{x \to \infty} \frac{\pi(x)}{\psi(x)/\log x} = 1$$

(from theorems (8.4) and (8.5)). Thus there is some $x_0$ such that

$$\left| \frac{\pi(x)}{\psi(x)/\log x} - 1 \right| < \epsilon.$$

Then due to the above arguments we know that for these $x$'s, there is a prime in $(x, cx)$.

(5) Define

$$\epsilon(c) = \frac{1}{2}\left(\frac{c - 1}{c + 1}\right).$$

Since $c > 1$ we know that $\epsilon(c) > 0$. From he definition of limit, we know that there is an $x_0(c)$ such that

$$1 - \epsilon(c) < \frac{\pi(x)}{x/\log x} < 1 + \epsilon(c) \qquad x > x_0(c).$$

Multiplying through by $x/\log x$ (which will be positive for $x > 1$, which is all of the $x$-values that we care about), we get that

$$(1 - \epsilon(c))\frac{x}{\log x} < \pi(x) < (1 + \epsilon(c))\frac{x}{\log x} \qquad x > x_0(x).$$

In order to show that there is at least one prime in $(x, cx)$, we simply need to show that $\pi(cx) - \pi(x) > 0$.

$$\begin{aligned}
\pi(cx) - \pi(x) &> (1 - \epsilon(c))\frac{cx}{\log cx} - (1 + \epsilon(c))\frac{x}{\log x} \\
&> ((c-1) - (c+1)\epsilon(c))\frac{x}{\log cx} \\
&= \left(\frac{c-1}{2}\right)\frac{x}{\log cx} \\
&> 0
\end{aligned}$$

as desired.