MATH 126 PROBLEM SET 4: BASIC RESULTS

This problem set is due Wednesday October 25. Except in questions 2) and 5) all vector spaces are assumed to be finite dimensional over an algebraicly closed field of characteristic 0. All representations are assumed to be representations of finite groups.

1) (This question takes you through the answer to a question one of you asked me. It would have better belonged on the previous problem set.) Let H be a subgroup of G. Show that if $\{g_i\}$ is a set of left coset representatives for $H\backslash G$ the $\{g_i^{-1}\}$ is a set of right coset representatives for G/H.

If (τ, W) is a representation of H let W_0 denote the set of $f \in \mathrm{Ind}_H^G W$ such that $f(g) = 0$ for all $g \in G - H$. Show that W_0 is an H-subrepresentation of $\mathrm{Ind}_H^G W$ which is isomorphic to W via the map $f \mapsto f(1)$. Show also that as vector spaces $\mathrm{Ind}_H^G W = \oplus_i g_i^{-1} W_0$.

Now suppose that (ρ, V) is a representation of G and that $W \subset V$ is an H-subrepresentation such that $V = \oplus_i g_i^{-1} W$, as vector spaces. Show that $V \cong \mathrm{Ind}_H^G W$ as representations of G. [Hint: consider the map $f \mapsto \sum_i g_i^{-1} f(g_i)$.]

2) Suppose that p is a prime number. Exhibit a representation V of $\mathbb{Z}/p\mathbb{Z}$ over the field $\mathbb{F}_p$ and a subrepresentation $W \subset V$ for which there is no subrepresentation $U \subset V$ with $V = W \oplus U$. Is your representation V isomorphic to the direct sum of irreducible representations?

3) Suppose a linear map $\alpha : V \to V$ satisfies $\alpha^n = 1$ for some positive integer n. Define a representation $\rho : \mathbb{Z}/n\mathbb{Z} \to GL(V)$ by $\rho(1) = \alpha$. Using the fact that ρ is a direct sum of irreducibles, show that α is diagonalisable.

4) Show that if ρ is a two dimensional representation and $\mathrm{Im}\,\rho$ contains two non-commuting matrices then ρ is irreducible.

5) Let $\rho : \mathbb{Z}/3\mathbb{Z} \to GL_2(\mathbb{R})$ be the representation defined by $\rho(1) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. Show that ρ is an irreducible representation. Find a non-scalar two by two matrix α which commutes with $\rho(g)$ for all $g \in \mathbb{Z}/3\mathbb{Z}$.

6) Let ρ be an irreducible representation of G. If z lies in the center of G show that $\rho(z)$ is a scalar matrix. Deduce that every irreducible representation of an abelian group is one dimensional.

7) Let $Q = \langle i, j : i^4 = i^2 j^2 = ijij^3 = 1 \rangle$. Recall that you showed that Q has order 8 and that it has an irreducible 2-dimensional representation. How many irreducible representations does Q have?

8) Show that

$$R(Q) \cong \mathbb{Z}[x, y, z]/(x^2 - 1, y^2 - 1, (x-1)z, (y-1)z, z^2 - (x+1)(y+1)).$$