MATH 126 PROBLEM SET 5: CHARACTER THEORY

This problem set is due Wednesday November 1. All groups are assumed to be finite and all vector spaces are assumed to be finite dimensional over an algebraicly closed field of characteristic 0.

1) If $A \in GL_n(\mathbb{C})$ satisfies $A^N = 1$ for some positive integer N show that all eigenvalues of A are roots of unity and hence that $\operatorname{tr}(A^{-1})$ is the complex conjugate of $\operatorname{tr} A$. Deduce that if V is a representation of a finite group G then $\chi_V^\vee$ is the complex conjugate of χ_V.

2) Suppose that (ρ, V) and (τ, W) are non-isomorphic irreducible representations of G. Show that $\operatorname{tr} \rho(\chi_V) = 1$ and deduce that $\rho((\dim V)\chi_V)$ is the identity endomorphism of V. Also show that $\operatorname{tr} \tau(\chi_V) = 0$ and deduce that $\tau((\dim V)\chi_V) = 0$.

Hence show that if (σ, U) is any representation of G that $\sigma((\dim V)\chi_V)$ is an element of $\operatorname{End}_G(U)$ which is also a projection operator with image $\operatorname{Hom}_G(V, U) \otimes V \subset U$.

3) If (ρ, V) is an irreducible n-dimensional representation of G and if $z \in Z(G)$ show that $\chi_V(z) = (\dim V)\zeta$ for some root of unity ζ. Deduce that

$$(\chi_V, \chi_V) \geq (\dim V)^2 \# Z(G)/\# G,$$

and hence that $(\dim V)^2 \leq \# G/\# Z(G)$.

4) Suppose G has irreducible representations $V_1, ..., V_h$ and conjugacy classes $[g_1], ..., [g_h]$. Let A denote the $h \times h$-matrix with

$$A_{ij} = \#[g_j]\chi_{V_i}(g_j),$$

and B denote the $h \times h$-matrix with

$$B_{ij} = \chi_{V_j}^\vee(g_i).$$

Show that $AB = (\# G)1_h$ and then that $BA = (\# G)1_h$.

Deduce that

$$\sum_{i=1}^{h} \chi_{V_i}^\vee(g)\chi_{V_i}(h)$$

is zero if $[g] \neq [h]$, and is $\# G/\#[g]$ if $[g] = [h]$.

5) In this question we will let $k(t)$ denote the field of rational functions in one variable t over k and $k((t))$ the field of formal Laurent series

$$\sum_{n=N}^{\infty} a_n t^n$$

where N is any integer. Recall that the is a natural embedding $k(t) \hookrightarrow k((t))$ which sends a rational function to its Laurent expansion at 0.

Let V be a faithful representation of G and W an irreducible representation of G. Set

$$f(t) = \sum_{n=0}^{\infty} (\chi_W, \chi_V^n) t^n \in k((t)).$$

Show that

$$f(t) = (1/\#G) \sum_{g \in G} \chi_W^\vee(g)/(1 - \chi_V(g)t) \in k(t).$$

Use the fact that V is faithful to show that $\chi_V(g) = \dim V$ if and only if g is the identity in G. Deduce that $f(t) \neq 0$ and hence that W is an irreducible constituent of $V^{\otimes n} = V \otimes ... \otimes V$ for some non-negative integer n.

6) Let V be the representation of S_4 which is induced from the one dimensional representation of $\langle(1234)\rangle$ which sends (1234) to $\sqrt{-1}$. Decompose V into irreducibles.

7) Find the character table of the alternating group A_5. [For instance you might like to start by finding irreducible representations of dimensions 1 and 4. You could the try to decompose the symmetric and alternating square of this 4 dimensional representation. Question 4) may help.]

6) Let G be a finite group. Consider the ring R of polynomials over $\mathbb{C}$ in $\#G$ variables X_g for $g \in G$. Let A denote the $\#G \times \#G$-matrix with rows and columns labeled by elements of G and with $X_{gh^{-1}}$ in the (g,h)-position. Set $P_G(X) = \det A$, a homogeneous polynomial in R of degree $\#G$. Let h denote the number of conjugacy classes in G. Show that there are homogeneous polynomials $P_1, ..., P_h \in R$ with degrees $d_1, ..., d_h$ such that

$$P_G = \prod_{i=1}^{h} P_i^{d_i}.$$

[Hint: Show that A is a matrix for $\sum_{g \in G} X_g g$ acting on the regular representation of G.]

Calculate P_{S_3}. [You may leave it as a product polynomials.]

(I've been told that Frobenius invented representation theory to explain this factorisation which he had noticed for small groups G.)