Math 126 Exercise set 4.

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1. We can make the cyclic group C_n into a graph by connecting each element k to both $k+1$ and $k-1$ as k ranges over the integers mod n. This is the graph of the regular n-gon. What are the eigenvalues of the adjacency matrix of this graph and what are the multiplicities. [Hint: the group C_n is abelian, and so each irreducible representation is one dimensional, and hence all characters are eigenvectors for the adjacency matrix.]

The Petersen graph is a famous graph in graph theory. In fact, there is a whole book devoted to the study of this graph. The Petersen graph has 10 vertices, and each vertex is is connected to exactly three other vertices by the following rule: Label the vertices as the two element subsets of $\{1,2,3,4,5\}$. Each two element subset is connected to those two element subsets which have empty intersection with it. For example, $\{1,2\}$ is connected to $\{3,4\}$, $\{3.5\}$, and $\{4,5\}$. Let V denote the set of vertices of the Petersen graph, and A denote its adjacency matrix.

2. What are the eigenvalues and multiplicities of A? [HInt: In class we showed that representation of S_5 on $\mathcal{F}(V)$ decomposes into the sum of the trivial representation (call it $\mathbf{1}$) a four dimensional irreducible representation (call it $\mathbf{4}$) and a five dimensional irreducible representation (call it $\mathbf{5}$). The subspace $\mathbf{1} \oplus \mathbf{4}$ was obtained as the image of the space $\mathcal{F}(\{1,2,3,4,5\})$ under the map

$$T : \mathcal{F}(\{1,2,3,4,5\}) \to \mathcal{F}(V), \quad (Tf)(\{a,b\}) = f(a) + f(b).$$

The space $\mathcal{F}(\{1,2,3,4,5\})$ itself decomposes as

$$\mathcal{F}(\{1,2,3,4,5\}) = \mathbf{1} \oplus \mathbf{4}$$

where $\mathbf{1}$ consists of the constants, and $\mathbf{4}$ consists of those functions which satisfy

$$f(1) + f(2) + f(3) + f(4) + f(5) = 0.$$

The image $T(\mathbf{1})$ of $\mathbf{1}$ under T consists of the constants in $\mathcal{F}(V)$, and a constant in $\mathcal{F}(V)$ is an eigenvector of A with eigenvalue 3. We also expect (actually know) that every Tf where f satisfies the above equation must be an eigenvector, i.e. that $ATf = T(\lambda f) = \lambda(Tf)$ for a suitable λ. Find λ. This is enough to determine the eigenvalue of A on the five dimensional irreducible subspace of $\mathcal{F}(V)$. Why, and what is it?]

We have found a five dimensional irreducible representation of S_6 by decomposing $\mathcal{F}(\{1,2,3,4,5,6\})$ into the constants (the trivial representation) and the five dimensional subspace consisting of all functions which satisfy $f(1)+f(2)+f(3)+f(4)+f(5)+f(6)=0$.

3. Let X be the space of all two element subsets of $\{1,2,3,4,5,6\}$ so $\#X = 15$. There is a map $T : \mathcal{F}(\{1,2,3,4,5,6\}) \to \mathcal{F}(X)$ just as in the case studied above, so there is a nine dimensional invariant complement. Compute the character of S_6 on this nine dimensional space and show that it is irreducible.

4. Let M be the set of ordered pairs of three element subsets of $\{1,2,3,4,5,6\}$ so $\#M = 20$. Compute the character of S_6 on $\mathcal{F}(M)$ and show that this representation contains the one, five and nine dimensional irreducibles described above, and that the complementary five dimensional invariant subspace is irreducible.