Review: Orthogonality relations for matrix elements of irreducible representations.

For any two functions on G we defined their scalar product by

$$(f_1, f_2) = (1/\#G) \sum_{a \in G} f_1(a)\overline{f_2(a)}.$$

Then

If $r^1 \not\sim r^2$, then $(r^2_{kl}, r^1_{ji}) = 0$ for all i, j, k and l. (3.3)

In other words, matrix entries from two inequivalent representations are orthogonal.

For a single irreducible $r = r^1 = r^2$ we have

$$(r^2_{kl}, r^1_{ij}) = (1/n)\delta_{ki}\delta_{lj} \quad (3.4)$$

so that distinct matrix entries from the same irreducible unitary representation are orthogonal, and each matrix entry has length $1/n^{\frac{1}{2}}$.

Characters

Let r be a representation of the group G on the vector space V. The dimension of V is called the *degree* of the representation, r. The *character* of the representation r is the function χ^r defined on G by the formula

$$\chi^r(a) = \operatorname{tr} r(a) = \sum_i r_{ii}(a). \tag{4.1}$$

If we take $a = e$, so that $r(e)$ is the identity operator, whose trace is dim V, we see that

$$\chi^r(e) = \dim V. \tag{4.2}$$

For any two linear transformations we have $\operatorname{tr} AB = \operatorname{tr} BA$; so, if B is non-singular, $\operatorname{tr} BAB^{-1} = \operatorname{tr} A$. Thus,

$$\chi(bab^{-1}) = \chi(a) \tag{4.3}$$

if χ is the character of any representation. In other words, χ is a function which is constant on conjugacy classes. Such a function is called a *central* function.

For any representation r, we can introduce a Hermitian scalar product which is invariant under $r(a)$ for all $a \in G$. This means that if we take adjoints with respect to this scalar product, we have $r(a)^* = r(a^{-1})$. But $\operatorname{tr} r(a)^*$ is the complex conjugate of $\operatorname{tr} r(a)$, so

$$\chi(a^{-1}) = \overline{\chi(a)}. \tag{4.4}$$

Orthogonality relations for characters

Let r^1 and r^2 be irreducible representations of G.

The character χ^1 of the representation r^1 is given by

$$\chi^1 = \sum_i r^1_{ii}$$

and similarly for the character χ^2 of r^2. It now follows from (3.3) and (3.2) that

$$\text{if } r^1 \not\sim r^2, \quad \text{then } (\chi^1, \chi^2) = 0, \tag{4.6}$$

and

$$(\chi, \chi) = 1, \text{ if } \chi \text{ is the character of an irreducible representation.} \tag{4.7}$$

Characters of direct sums.

Let r^1 and r^2 be representations of G. Then it follows from the matrix form of $r^1 \oplus r^2$ that

$$\chi^{r^1 \oplus r^2} = \chi^{r^1} + \chi^{r^2}. \qquad (4.5)$$

Now let r be a representation of G on a vector space, V, which is not necessarily irreducible, and let

$$r = r^1 \oplus \cdots \oplus r^k$$

be a decomposition of r into irreducible representations. Let ϕ be the character of r, and let χ_i be the character of r^i, so that

$$\phi = \chi_1 + \cdots + \chi_k.$$

Let s be some particular irreducible representation of G and let χ be its character. Then

$$(\phi, \chi) = (\chi_1, \chi) + \cdots + (\chi_k, \chi).$$

The terms on the right are all zero or one, according as $r^i \nsim r$ or $r^i \sim r$. Thus,

> (ϕ, χ) is the number of terms in the decomposition of r which are isomorphic to s. In particular, this number does not depend on the particular choice of decomposition. (4.8)

From (4.8) it follows that *two representations with the same character are equivalent.*

□ is irreducible if and only if (□ , □)=1

Proof: any character ϕ can be written as

$$\phi = m_1\chi_1 + \cdots + m_p\chi_p,$$

where the χ_i are irreducible orthogonal characters. Hence

$$(\phi, \phi) = m_1^2 + \cdots + m_p^2 \qquad (4.9)$$

We already know that if □ is irreducible then (□ , □)=1. Conversely, If (□ , □)=1, then the only way that (4.9) could hold is for all the m_i but one =0, and one of them =1 which says that □ is irreducible.

dim $\mathrm{Hom}_G(W,V)$ when V is irreducible

Let ϕ be the character of a representation of G on a vector space W, and let χ be the character of an irreducible representation of G on the vector space V. If we decompose

$$W = W_1 \oplus \cdots \oplus W_k$$

into irreducibles, we see that

$$\mathrm{Hom}_G(W, V) = \mathrm{Hom}_G(W_1, V) \oplus \cdots \oplus \mathrm{Hom}_G(W_k, V).$$

By Schur's lemma, each of these spaces is either one dimensional or zero dimensional according to whether the representation of G on W_i is or is not equivalent to the representation of G on V. Combining this with (4.8) we see that

$$(\phi, \chi) = \dim \mathrm{Hom}_G(W, V). \tag{4.10}$$

dim $\mathrm{Hom}_G(U,V)$ in general.

Now let r_u and r_v be representations of G on U and V. We do not assume that r_u and r_v are irreducible. We wish to compute $\dim \mathrm{Hom}_G(U, V)$.

Let us first consider a special case. Suppose $U = V = W \oplus W$, where W is irreducible. We can write any vector in U as $(\mathbf{w}_1, \mathbf{w}_2)$, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are in W. Thus, for any $T \in \mathrm{Hom}(V, V)$ we have

$$T(\mathbf{w}_1, \mathbf{w}_2) = (T_{11}\mathbf{w}_1 + T_{12}\mathbf{w}_2, T_{21}\mathbf{w}_1 + T_{22}\mathbf{w}_2)$$

where $T_{ij} \in \mathrm{Hom}(W, W)$. So

$$\begin{aligned} T \circ r_{W\oplus W}(a)(\mathbf{w}_1, \mathbf{w}_2) &= T(r_W(a)\mathbf{w}_1, r_W(a)\mathbf{w}_2) \\ &= (T_{11}r_W(a)\mathbf{w}_1 + T_{12}r_W(a)\mathbf{w}_2, T_{21}r_W(a)\mathbf{w}_1 + T_{22}r_W(a)\mathbf{w}_2) \end{aligned}$$

while

$$\begin{aligned} &r_{W\oplus W}(a)T(\mathbf{w}_1, \mathbf{w}_2) \\ &= (r_W(a)(T_{11}\mathbf{w}_1 + T_{12}\mathbf{w}_2), r_W(a)(T_{21}\mathbf{w}_1 + T_{22}\mathbf{w}_2)) \\ &= (r_W(a)T_{11}\mathbf{w}_1 + r_W(a)T_{12}\mathbf{w}_2, r_W(a)T_{21}\mathbf{w}_1 + r_W(a)T_{22}\mathbf{w}_2). \end{aligned}$$

So $T \in \mathrm{Hom}_G(V, V)$ if and only if each $T_{ij} \in \mathrm{Hom}_G(W, W)$. By Schur's lemma, each T_{ij} ranges over a one-dimensional space, hence $\dim \mathrm{Hom}_G(W \oplus W, W \oplus W) = 4 = 2 \times 2$.

dim $\mathrm{Hom}_G(U,V)$ in general, continued

For any representation, we may make the decomposition

$$U = (U_1 \oplus \cdots \oplus U_{p_1}) \oplus (U_{p_1+1} \oplus \cdots \oplus U_{p_1+p_2}) \oplus \cdots (\cdots U_{p_1+\cdots p_k})$$

where the first p_1 spaces are all equivalent to the irreducible representation W_1, the next p_2 spaces are all equivalent to the irreducible representation W_2 etc., and $W_1, \ldots, W_k$ are *inequivalent* irreducible representations of G. We may make the same decomposition

$$V = (V_1 \oplus \cdots \oplus V_{q_1}) \oplus (V_{q_1+1} \oplus \cdots \oplus V_{q_1+q_2}) \oplus \cdots$$

for V. By Schur's lemma, any $T \in \mathrm{Hom}_G(U, V)$ when applied to any $\mathbf{u} \in U_1 \oplus \cdots \oplus U_{p_1}$ must give $T\mathbf{u}$ lying in $V_1 \oplus \cdots \oplus V_{q_1}$. Then the same argument as in the special case shows that

$$\dim \mathrm{Hom}_G(U, V) = p_1 q_1 + p_2 q_2 + \cdots + p_k q_k. \tag{4.11}$$

In particular, if $U = V$,

$$\dim \mathrm{Hom}_G(V, V) = p_1^2 + \cdots + p_k^2,$$

where p_i is the number of times that the ith irreducible representation occurs in V.

The representation on function spaces induced from an action.

Suppose that we are given an action of the group G on the set M. Let $\mathscr{F}(M)$ denote the vector space of all complex-valued functions on M. Define an action of G on $\mathscr{F}(M)$ by

$$(af)(x) = f(a^{-1}x).$$

Put another way, we define af by

$$af = f \circ a^{-1},$$

We have

$$a(bf) = (bf) \circ a^{-1} = (f \circ b^{-1}) \circ a^{-1} = f \circ (b^{-1} \circ a^{-1}) = f \circ (ab)^{-1} = (ab)f.$$

So we get a representation of G on $\mathscr{F}(M)$

Let us denote this representation by r^M .

The character fixed point formula.

Introduce the following basis of $\mathcal{F}(M)$:

Let δ_x be the function on M defined by

$$\delta_x(y) = \begin{matrix} 1 & \text{if} & y = x \\ 0 & \text{if} & y \neq x \end{matrix}.$$

Then

$$(a\delta_x)(y) = \delta_x(a^{-1}y) = \begin{matrix} 1 & \text{if} & a^{-1}y = x, \text{ i.e. } y = ax \\ 0 & \text{if} & a^{-1}y \neq x, \text{ i.e. } y \neq ax \end{matrix}$$

so that

$$a\delta_x = \delta_{ax}.$$

The functions δ_x are clearly independent and they span $\mathcal{F}(M)$ since any function, f, can be written as $f = \sum_{x \in M} f(x)\delta_x$. For this basis, the diagonal elements of $r^M(a)$ will be one or zero according as $ax = x$ or $ax \neq x$. Thus

$$\chi^{r^M}(a) = \sum_{ax = x} 1 = \#(\text{fixed points of } a) \tag{5.1}$$

Orbits and direct sums

Suppose that M decomposes into orbits under the action of G:

$$M = M_1 \cup \cdots \cup M_k.$$

Then we have a corresponding decomposition

$$r^M = r^{M_1} \oplus \cdots \oplus r^{M_k}.$$

Indeed, we can identify r^{M_i} with the subrepresentation of r^M given by the action of G on functions which vanish outside of M_i. Any function on M can be written uniquely as

$f = f_1 + \cdots + f_k$, where each f_j vanishes outside of M_i.

Morphisms between function spaces.

Suppose that G acts on the two finite sets M and N. We wish to studv the space $\mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N))$ and the action of G on it. Notice that

$$\begin{aligned}\dim \mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N)) &= \dim \mathscr{F}(M) \times \dim \mathscr{F}(N)\\ &= (\# M)\cdot(\# N)\\ &= \#(N \times M)\\ &= \dim \mathscr{F}(N \times M).\end{aligned}$$

We claim that there is a natural isomorphism between $\mathscr{F}(N \times M)$ and $\mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N))$. Indeed, given any function K on $N \times M$ define the operator $T_K: \mathscr{F}(M) \to \mathscr{F}(N)$ by

$$(T_K f)(y) = \sum_{x \in M} K(y, x) f(x).$$

The map sending K into T_K is one-to-one; indeed, for any $u \in M$

$$(T_K \delta_u)(y) = K(y, u)$$

so if $T_K = 0$, then $K(y, u) = 0$ for all y and u, i.e. $K = 0$. Since $\mathscr{F}(N \times M)$ and $\mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N))$ have the same dimension, we conclude that the map sending K to T_K is an isomorphism of vector spaces.

Morphisms between function spaces, continued.

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The group G acts on both $\mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N))$ and on $\mathscr{F}(N \times M)$. We claim that the isomorphism just described is a G morphism, i.e. that the representations of G on these two spaces are equivalent. Indeed, letting r^{Hom} denote the representation of G on $\mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N))$, we have

$$r^{\mathrm{Hom}}(a) T_K = r^N(a) T_K r^M(a)^{-1}.$$

Morphisms between function spaces, continued.

$$r^{\mathrm{Hom}}(a)T_K = r^N(a)T_K r^M(a)^{-1}.$$

Now

$$r^M(a)^{-1}f(x) = f(ax)$$

so

$$(T_K r^M(a)^{-1}f)(y) = \sum K(y,x)f(ax)$$

and

$$\begin{aligned}(r^{\mathrm{Hom}}(a)T_K f)(y) &= \sum K(a^{-1}y,x)f(ax)\\ &= \sum K(a^{-1}y,a^{-1}x)f(x).\end{aligned}$$

But

$$(r^{N\times M}(a)K)(y,x) = K(a^{-1}y,a^{-1}x)$$

So

$$r^{\mathrm{Hom}}(a)T_K = T_{r^{N\times M}(a)K}$$

which was to be proved.

Morphisms between function spaces, continued.

We have proved that $r^{\mathrm{Hom}}(a)T_K = r^N(a)T_K r^M(a)^{-1}$ and we have

$$(r^{N\times M}(a)K)(y,x) = K(a^{-1}y, a^{-1}x)$$

Now $\mathrm{Hom}_G(\mathscr{F}(M), \mathscr{F}(N))$ is the space of elements in $\mathrm{Hom}(\mathscr{F}(M), \mathscr{F}(N))$ which satisfy

$$r^{\mathrm{Hom}}(a)T = T$$

for all $a \in G$. If $T = T_K$, the preceding equation shows that

$$K(a^{-1}y, a^{-1}x) = K(y,x)$$

for all $a \in G$. In other words, the function K must be constant on orbits of G on $N \times M$. Thus

$$\dim \mathrm{Hom}_G(\mathscr{F}(M), \mathscr{F}(N)) = \#\text{ of } G \text{ orbits on } N \times M. \tag{5.2}$$

In particular, on taking $M = N$ we see that

$$\begin{aligned}\dim \mathrm{Hom}_G(\mathscr{F}(M), \mathscr{F}(M)) &= \#\text{ of } G \text{ orbits on } M \times M \\ &= p_1^2 + \cdots + p_k^2 \end{aligned} \tag{5.3}$$

where p_i is the number of times that the ith irreducible representation of G occurs in $\mathscr{F}(M)$.

Example: S_n acting on {1,….,n}.

Consider the group S_n acting on the n-element set $M = \{1, \ldots, n\}$. On $M \times M$ there are two orbits

$$\{(x, y) | x \neq y\} \quad \text{and} \quad \{(x, x)\}.$$

Indeed, if $x \neq y$ and $z \neq w$ we can find a permutation σ such that $\sigma(x) = z$ and $\sigma(y) = w$ Thus, the set $\{(x, y) | x \neq y\}$ is a single orbit in $M \times M$. Similarly the set $\{(x, x)\}$ is a single orbit. Thus,

$$\dim \mathrm{Hom}_G(\mathscr{F}(M), \mathscr{F}(M)) = 2 = p_1^2 + \cdots + p_k^2$$

so $k = 2$ and $p_1 = p_2 = 1$. Thus, $\mathscr{F}(M)$ is the direct sum of two irreducible representations. We already know one of them – the trivial one-dimensional representation, corresponding to the constant functions. The other must then be $n - 1$ dimensional. Thus

$$\mathscr{F}(M) = \underset{\substack{\uparrow \\ \text{one} \\ \text{dimensional}}}{V_1} + \underset{\substack{\uparrow \\ n-1 \\ \text{dimensional}}}{V_2}$$

The regular representation.

We apply the results of the preceding section to the special case where $M = G$ and G acts on itself by left multiplication. The corresponding representation, r^G, of G on $\mathscr{F}(G)$ is called the *regular* representation. It is defined by $[r^G(a)f](b) = f(a^{-1}b)$. We have

$$\#G = \dim \mathscr{F}(G) = \sum p_i n_i$$

where p_i is the number of times that the ith irreducible representation occurs in $\mathscr{F}(G)$, while n_i is the dimension of the ith irreducible representation. Also

$$\begin{aligned}\dim \operatorname{Hom}_G(\mathscr{F}(G), \mathscr{F}(G)) &= \sum p_i^2 \\ &= \# \text{ of } G \text{ orbits on } G \times G.\end{aligned}$$

We compute the number of orbits as follows: we can always act on (a, b) by a^{-1} to get $(e, a^{-1}b)$. Thus, each orbit of G in $G \times G$ contains a point of the form (e, c). But this is the only element of this form in its orbit, since multiplying by d sends (e, c) into (d, dc). Thus each orbit contains a unique representative of the form (e, c), and hence the number of orbits is equal to $\#G$. Thus

$$\#G = \sum p_i^2.$$

Since $\sum p_i^2 = \sum p_i n_i$ we are led to guess that $p_i = n_i$, i.e. that each irreducible representation, W, occurs in $\mathscr{F}(G)$ with a multiplicity equal to its dimension, i.e. that

$$\dim \operatorname{Hom}_G(W, \mathscr{F}(G)) = \dim W. \tag{6.1}$$

The regular representation, continued.

We wish to prove that $\dim \operatorname{Hom}_G(W, \mathscr{F}(G)) = \dim W.$ (6.1)

We shall prove this fact by constructing an isomorphism between W^*, the dual space of W, and $\operatorname{Hom}_G(W, \mathscr{F}(G))$. To each $l \in W^*$ and to each $\mathbf{w} \in W$ we assign the function $f^l_{\mathbf{w}}$ on G defined by

$$f^l_{\mathbf{w}}(a) = \langle r(a^{-1})\mathbf{w}, l \rangle.$$

Here $r(a^{-1})\mathbf{w} \in W$ and $l \in W^*$, and $\langle \mathbf{v}, l \rangle$ denotes the value of the linear function $l \in W^*$ at the element $\mathbf{v}$ of V. For fixed l the map sending $\mathbf{w}$ into $f^l_{\mathbf{w}}$ is a map from W to $\mathscr{F}(G)$. Thus each $l \in V^*$ defines an element of $\operatorname{Hom}(W, \mathscr{F}(G))$. We must show that this element lies in $\operatorname{Hom}_G(W, \mathscr{F}(G))$, i.e. that

$$f^l_{r(b)\mathbf{w}} = r^G(b) f^l_{\mathbf{w}}$$

or that

$$f^l_{r(b)\mathbf{w}}(a) = f^l_{\mathbf{w}}(b^{-1}a) \quad \text{for all } a, b \in G.$$

But

$$\begin{aligned} f^l_{r(b)\mathbf{w}}(a) &= \langle r(a)^{-1} r(b) w, l \rangle \\ &= \langle r(a^{-1}b) w, l \rangle \\ &= \langle r(b^{-1}a)^{-1} w, l \rangle \\ &= f^l_{\mathbf{w}}(b^{-1}a) \end{aligned}$$

as required.

The regular representation, continued

Conclusion of the proof that

$$\dim \operatorname{Hom}_G(W, \mathscr{F}(G)) = \dim W. \tag{6.1}$$

We defined the G morphism from W* to $\mathscr{F}(G)$ by

$$f_{\mathbf{w}}^{l}(a) = \langle r(a^{-1})\mathbf{w}, l \rangle.$$

Furthermore, $f_{\mathbf{w}}^{l}(e) = \langle \mathbf{w}, l \rangle$ cannot be zero for all $\mathbf{w}$ unless $l = 0$. Thus the map of W^* into $\operatorname{Hom}_G(W, \mathscr{F}(G))$ that we have defined is injective. It follows that

$$p_i = \dim \operatorname{Hom}_G(W, \mathscr{F}(G)) \geqslant \dim W^* = n_i.$$

But it now follows from the two equations

$$\#G = \sum p_i n_i = \sum p_i^2$$

that we must have $p_i = n_i$ so (6.1) holds. Thus

$$\#G = \sum n_i^2. \tag{6.2}$$

Equations (6.1) and (6.2) have the following useful corollary. Suppose that we have found inequivalent irreducible representations $(r_1, W_1) \cdots (r_k, W_k)$ of G, with $\dim W_i = n_i$, such that $\sum_i^d n_i^2 = \#G$. Then it follows from (6.2) that there can be no other

Irreducible representation, i.e. ones not equivalent to the ones we have on our list.