Math 126 Lecture 4.

Basic facts in representation theory.

Notice.

Until further notice, all groups will be finite, and all vector spaces will be finite dimensional vector spaces over the complex numbers. A typical group will be denoted by G or H, the identity element by e and vector spaces by V, W, etc.

Definition of a representation of a group.

A **representation** r of G on V is an action of G on V in which every element acts as a linear transformation. So r is a homomorphism of G into the group $Gl(V)$ of all invertible linear transformations of V. We will sometimes write $r(a)$ for the image of a under r, so

$$r(a)v$$

is the image of $v \in V$ under the action of the linear transformation $r(a)$; or, if r is clear from the context we may write

$$av$$

instead of $r(a)v$.

The theory of group representations is the creation of Frobenius:

Georg Frobenius

lived from 1849 to 1917

Frobenius combined results from the theory of algebraic equations, geometry, and number theory,
which led him to the study of abstract groups, the representation theory of groups and the character theory of groups.

Find out more at:
http://www-history.mcs.st-andrews.ac.uk/history/
Mathematicians/Frobenius.html

Matrix form of a representation.

If we choose a basis of V then we will write the matrix associated to $r(a)$ by this basis as

$$(r_{ij}(a)).$$

The matrices $(r_{ij}(a))$ depend on the choice of basis, and so are determined only up to conjugacy. The fact that r is a representation, so

$$r(e) = I$$

where I is the identity operator, and

$$r(ab) = r(a)r(b)$$

translates into

$$r_{ij}(e) = \delta_{ij} \quad \text{and} \quad r_{ij}(ab) = \sum_k r_{ik}(a)r_{kj}(b).$$

Equivalence of two representations.

Let r and r' be representations of the same group G on vector spaces V and V'. We say that r and r' are **similar** or **equivalent** if there is a bijective (i.e. one to one and onto) linear map $T : V \to V'$ which is a morphism, i.e. such that

$$r'(a)T = Tr(a) \quad \forall \;\; a \in G.$$

The space of all $T \in \mathrm{Hom}(V, V')$ which satisfy the above equation is denoted by

$$\mathrm{Hom}_G(V, V').$$

Invariant subspaces.

A subspace of V is **invariant** if $r(a)W \subset W$ for all $a \in G$. Applying $r(a^{-1})$ we see that this is equivalent to $r(a)W = W$ for all $a \in G$. The restriction of $r(a)$ to W is then a representation of G on W called a **subrepresentation** of r.

Irreducible representations.

Clearly V itself and the subspace $\{0\}$ are invariant subspaces. If these are the only irreducible subspaces the representation r is called **irreducible**.

One dimensional representations.

If V is one dimensional, then r is irreducible since there is no room for any subspace to lie strictly between $\{0\}$ and V. A linear transformation on one dimensional space is just multiplication by a scalar. So a one dimensional representation of G is a map

$$\kappa : G \to \mathbb{C}$$

such that

$$\kappa(ab) = \kappa(a)\kappa(b), \qquad \kappa(e) = 1.$$

Since $a^{\#G} = e$, we see that $\kappa(a)$ is root of unity whose order is some divisor of $\#G$.

Representations of cyclic groups.

If $G = C_n$, and let b be a generator of C_n. For each integer k with $0 \leq k < n$ we may define

$$\kappa_k(b) = e^{2\pi ik/n} \quad \text{so} \quad \kappa_k(b^s) = e^{2\pi isk/n}.$$

These are inequivalent irreducible representations of C_n and in fact all of them (as we shall prove), and play a central role in the Fourier transform.

In fact, if G is commutative, all non trivial irreducible representations are one dimensional: Choose $a \in G$. The operator $r(a)$ has at least one eigenvalue, call it $\lambda(a)$. The set W of all $v \in V$ which are eigenvalues of a with eigenvalue $\lambda(a)$ is an invariant subspace. Proof: If $w \in W$ then $r(b)w \in W$ for all $b \in G$ since

$$r(a)r(b)w = r(ab))w = r(ba)w = r(b)r(a)w = r(b)\lambda(a)w = \lambda(a)r(b)w.$$

Since $w =\neq \{0\}$ we must have $W = V$ so $r(a) = \lambda(a)I$. Choose another $c \in G$. Again all $v \in V$ are eigenvectors for $r(c)$. Continuing, we see that every $a \in G$ is represented by a scalar operator $\lambda(a)I$. But then any subspace is invariant, so V must be one dimensional.

Direct sums.

let r be a representation of G on V and s a representation of G on W. Then $r \oplus s$ is the representation of G on $V \oplus W$ given by $a \mapsto r(a) \oplus s(a)$. In terms of a basis of V and basis of W which are combined to form a basis of $V \oplus W$ the matrix form of $r \oplus s$ is "block diagonal" with r_{ij} in the upper let corner and $sk\ell$ in the lower right corner.

Tensor product.

Similarly, we define the representation $r \otimes s$ on $V \otimes W$ as being

$$(r \otimes s)(a) = r(a) \otimes s(a).$$

If $e_1, \ldots, e_m$ is a basis of V and $f_1, \ldots, f_n$ is a basis of W then the $e_i \otimes f_k$ form a basis of $V \otimes W$ and the corresponding matrix form of $r \otimes s$ is

$$(r \otimes s)_{ik,j\ell}(a) = r_{ij}(a) s_{k\ell}(a).$$

For example, in the case of the one dimensional representations of C_n discussed above,

$$\kappa_k \otimes \kappa_\ell = \kappa_s \quad \text{where } s =\equiv k + \ell \bmod n.$$

Unitary representations.

Let $(\, , \,)$ be a scalar product on V. We say that $(\, , \,)$ is **invariant** under a representation r of G on V, or that r is **unitary** relative to $(\, , \,)$, if

$$(r(a)u, r(a)v) = (u, v) \quad \forall a \in G, \quad u, v, \in V.$$

Averaging over the group.

If we start with a scalar product $\langle\ ,\ \rangle$ which is not necessarily invariant we can produce an invariant one by defining

$$(u, v) = \frac{1}{\#G} \sum_{b \in G} \langle r(b)u, r(b)v \rangle.$$

This is clearly linear in u, anti-linear in v and $(u.u) > 0$ so is a scalar product. It is invariant since

$$(r(a)u, r(a)v)) = \frac{1}{\#G} \sum_{b \in G} \langle r(b)r(a)u, r(b)r(a)v \rangle = \frac{1}{\#G} \sum_{c \in G} \langle r(c)u, r(c)v \rangle = (u, v)$$

since summing over ab for fixed a is the same as summing over all $c \in G$.

Maschke's theorem.

This says that if W is an invariant subspace then it has an invariant complement. Proof: Choose an invariant scalar product. Then the orthogonal complement $W^{\perp}$ with respect to this scalar product is invariant.

In particular it follows that every representation is equivalent to a finite direct sum of irreducible representations.

Heinrich Maschke 1853 - 1908

Schur's lemma.

Let r and s be irreducible representations of G on V and W, and $T \in \mathrm{Hom}_G(V, W)$. Schur's lemma makes two assertions:

$$r \not\sim s \quad \Rightarrow T = 0, \tag{1}$$

and, if

$$r = s \quad (\text{so } V = W) \;\; \text{then } T = zI \tag{2}$$

for some scalar z.

Proof of (1): The subspace $\ker T \subset V$ is invariant. so the alternatives are $\ker T = V$ in which case $T = 0$, or $\ker T = \{0\}$, in which case T is one to one. Also $\mathrm{Im}\,(T) \subset W$ is invariant so $= \{0\}$ and so $T = 0$ or $\mathrm{Im} T = W$ is which case T is surjective so an equivalence contrary to hypothesis.

Proof of (2): Apply (1) to $T - cI$ where c is an eigenvalue of T. We must have $T - cI = 0$ or $T = cI$. $\square$

Issai Schur

Biography of Schur.

Issai Schur
Born: 10 Jan 1875 in Mogilyov, Mogilyov province, Russian Empire (now Belarus)
Died: 10 Jan 1941 in Tel Aviv, Palestine (now Israel)

Although Issai Schur was born in Mogilyov on the Dnieper, he spoke German without a trace of an accent, and nobody even guessed that it was not his first language. He went to Latvia at the age of 13 and there he attended the Gymnasium in Libau, now called Liepaja.

In 1894 Schur entered the University of Berlin to read mathematics and physics. Frobenius was one of his teachers and he was to greatly influence Schur and later to direct his doctoral studies. Frobenius and Burnside had been the two main founders of the theory of representations of groups as groups of matrices. This theory proved a very powerful tool in the study of groups and Schur was to learn the foundations of this subject from Frobenius. Schur then made major steps forward, both in work of his own and work done in collaboration with Frobenius.

In 1901 Schur obtained his doctorate with a thesis which examined rational representations of the general linear group over the complex field. Functions which Schur introduced in his thesis are today called S-functions, where the S stands for Schur

In 1903 Schur became a lecturer at Berlin University and then, from 1911 until 1916, he held a professorship in mathematics at the University of Bonn. He returned to Berlin in 1916 and there he built his famous school and spent most of the rest of his life there. He was promoted to full professor in Berlin in 1919, three years after he returned there, and he held this chair until he was dismissed by the Nazis in 1935.

Schur is mainly known for his fundamental work on the representation theory of groups but he also worked in number theory, analysis and other topics described below. Between 1904 and 1907 he worked on projective representations of groups and group characters. One of the most fundamental results which he discovered at this time is today called Schur's Lemma.

In a series of papers he introduced the concept now known as the 'Schur multiplier'. This is an extremely important abstract concept which arose from the concrete problems that Schur was studying. Much later, in 1949, Eilenberg and Mac Lane defined cohomology groups. They were unaware at that time that the second cohomology group with coefficients in the nonzero complex numbers is the Schur multiplier, and therefore that Schur had made some of the first steps forty years earlier.

Around 1914 Schur's interest in representations of groups was put to one side while he worked on other topics but, around 1925, developments in theoretical physics showed that groups representations were of fundamental importance in that subject. Schur returned to work on representation theory with renewed vigour and he was able to complete the programme of research begun in his doctoral dissertation and give a complete description of the rational representations of the general linear group.

Schur was also interested in reducibility, location of roots and the construction of the Galois group of classes of polynomials such as Laguerre and Hermite polynomials. In [1] an indication of the other topics that Schur worked on is given:-

First there was pure group theory, in which Schur adopted the surprising approach of proving without the aid of characters, theorems that had previously been demonstrated only by that means.

Second, he worked in the field of matrices.

Third, he handled algebraic equations, sometimes proceeding to the evaluation of roots, and sometimes treating the so-called equation without affect, that is, with symmetric Galois groups. He was also the first to give examples of equations with alternating Galois groups.

Fourth, he worked in number theory;

Fifth in divergent series;

Sixth in integral equations; and lastly

in function theory.

The school which Schur built at Berlin was of major importance not only for the representation theory of groups but, as indicated above, for other areas of mathematics. The school partly worked through the Schur's lecturing.

Among the students who completed their doctorates under Schur were Richard Brauer, Alfred Brauer (Richard Brauer's brother), Robert Frucht, Bernhard Neumann, Richard Rado, and Helmut Wielandt. There were others who worked under Schur such as Kurt Hirsch, Walter Ledermann, Hanna Neumann and Menahem Max Schiffer.

Ledermann in [Issai Schur and his school in Berlin, Bull. London Math. Soc. 15 (1983), 97-106.] describes Schur as a teacher:-

Schur was a superb lecturer. His lectures were meticulously prepared... [and] were exceedingly popular. I remember attending his algebra course which was held in a lecture theatre filled with about 400 students. Sometimes, when I had to be content with a seat at the back of the lecture theatre, I used a pair of opera glasses to get at least a glimpse of the speaker.

In 1922 Schur was elected to the Prussian Academy, proposed by Planck, the secretary of the Academy. Planck's address which listed Schur's outstanding achievements had been written by Frobenius, at least five years earlier, as Frobenius died in 1917.

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From 1933 events in Germany made Schur's life increasingly difficult. Hirsch spoke of the events of 1 April 1933 when posters carried the message 'Germans defend yourselves against jewish atrocity propaganda : buy only at German shops':-

That was the so-called 'Boycott Day', the day on which Jewish shops were boycotted and Jewish professors and lecturers were not allowed to enter the university. Everybody who was there had to make a little speech about the rejuvenation of Germany etc. And Bieberbach did this quite nicely and then he said 'A drop of remorse falls into my joy because my dear friend and colleague Schur is not allowed to be among us today'.

On 7 April 1933 the Nazis passed a law which, under clause three, ordered the retirement of civil servants who were not of Aryan descent, with exemptions for participants in World War I and pre-war officials. Schur had held an appointment before World War I which should have qualified him as a civil servant, but the facts were not allowed to get in the way, and he was 'retired'.

Schiffer wrote in M M Schiffer, Issai Schur : Some personal reminiscences, in H Begehr (ed.), Mathematik in Berlin : Geschichte und Dokumentation (Aachen, 1998).

When Schur's lectures were cancelled there was an outcry among the students and professors, for Schur was respected and very well liked. The next day Erhard Schmidt started his lecture with a protest against this dismissal and even Bieberbach, who later made himself a shameful reputation as a Nazi, came out in Schur's defence. Schur went on quietly with his work on algebra at home.

Schur saw himself as a German, not a Jew, and could not comprehend the persecution and humiliation he suffered under the Nazis. In fact Schur's dismissal was revoked and he was able to carry out some of his duties for a while. By November 1933 when Walter Ledermann took his Staatsexamen he was examined by Schur together with Bieberbach who was wearing a Nazi uniform.

There were invitations to Schur to go to the United States and to Britain but he declined them all, unable to understand how a German was not welcome in Germany. For example Ledermann obtained a scholarship to go to St Andrews in Scotland in the spring of 1934 and he tried unsuccessfully to persuade Schur to join him in St Andrews.

Schur continued to suffer the humiliation that was heaped on him.

Schur told me that the only person at the Mathematical Institute in Berlin who was kind to him was Grunsky, then a young lecturer. Long after the war, I talked to Grunsky about that remark and he literally started to cry: "You know what I did? I sent him a postcard to congratulate him on his sixtieth birthday. I admired him so much and was very respectful in that card. How lonely he must have been to remember such a small thing."

Later in 1935 Schur was dismissed from his chair in Berlin but he continued to work there suffering great hardship and difficulties. Alfred Brauer, the brother of Richard Brauer, writes:-

When Landau died in February 1938, Schur was supposed to give an address at his funeral. For that reason he was in need of some mathematical details from the literature. He asked me to help him in this matter. Of course I was not allowed to use the library of the mathematical institute which I had built up over many years. Finally I got an exemption for a week and could use the library of the Prussian Staatsbibliothek for a fee. ... So I could answer at least some of Schur's questions.

Pressure was put on Schur to resign from the Prussian Academy to which he had been honoured to be elected in 1922. On 29 March 1938 Bieberbach wrote below Schur's signature on a document of the Prussian Academy:-

> I find it surprising that Jews are still members of academic commissions.

Just over a week later, on 7 April 1938, Schur resigned from Commissions of the Academy. However, the pressure on him continued and later that year he resigned completely from the Academy.

Schur left Germany for Palestine in 1939, broken in mind and body, having the final humiliation of being forced to find a sponsor to pay the 'Reichs flight tax' to allow him to leave Germany. Without sufficient funds to live in Palestine he was forced to sell his beloved academic books to the Institute for Advanced Study in Princeton. He died two years later on his 66th birthday on a park bench in Tel Aviv.