Math 126 Lecture 10

The plactic monoid

Contents

New definition of a tableau.

Young diagrams are as before, for example

□	□	□	□
□	□	□	□
□	□	□	
□	□		

But now the word tableau is a choice of filling the diagram with positive integers where repetition is allowed, and the numbers are non-decreasing from left to right along rows, and strictly decreasing along columns. For example

1	2	2	3
2	3	5	5
4	4	6	
5	6		

is a tableau associated to the above diagram.

The word of a tableau.

1	2	2	3
2	3	5	5
4	4	6	
5	6		

The **word** associated to a tableau is obtained by reading the numbers from left to right starting with the bottom row. So the word of the above tableau is

$$5644623551223.$$

We can recover the tableau from its word by noting that the rows must end just before a strict decrease in the entry:

$$56|446|2355|1223$$

gives the row breaks. But not every word comes from a tableau: The sizes of the rows must be non-decreasing, and when stacked on top of one another the entries must be strictly decreasing from top down.

Insertion and bumping.

The **Schensted bumping algorithm** takes a tableau T and a positive integer n and produces a tableau whose diagram has one more box. It works as follows: If n is as least as large as the largest entry in the top row, simply add n in a new box at the right of the top row. If not, find the leftmost entry in the top row which is strictly larger than n, replace that entry m with n, and repeat the process with m on the second row and continue. For example, inserting 2 into the above tableau leads to the following steps:

$$\begin{array}{cccccc} 1 & 2 & 2 & 3 & \leftarrow & 2 \\ 2 & 3 & 5 & 5 & & \\ 4 & 4 & 6 & & & \\ 5 & 6 & & & & \end{array} \Rightarrow \begin{array}{cccccc} 1 & 2 & 2 & 2 & & \\ 2 & 3 & 5 & 5 & \leftarrow & 3 \\ 4 & 4 & 6 & & & \\ 5 & 6 & & & & \end{array}$$

$$\Rightarrow \begin{array}{cccccc} 1 & 2 & 2 & 2 & & \\ 2 & 3 & 3 & 5 & & \\ 4 & 4 & 6 & & \leftarrow & 5 \\ 5 & 6 & & & & \end{array} \Rightarrow \begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 3 & 3 & 5 \\ 4 & 4 & 5 & \\ 5 & 6 & 6 & \end{array} .$$

Again

The final result of inserting 2 into the above tableau is to effect the transformation

$$
\begin{array}{cccc}
1 & 2 & 2 & 3 \\
2 & 3 & 5 & 5 \\
4 & 4 & 6 & \\
5 & 6 & &
\end{array}
\quad \Rightarrow \quad
\begin{array}{cccc}
1 & 2 & 2 & 2 \\
2 & 3 & 3 & 5 \\
4 & 4 & 5 & \\
5 & 6 & 6 &
\end{array}
\quad .
$$

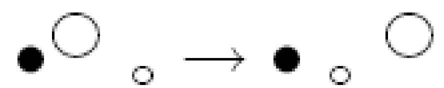
Repeat:

$$
\begin{array}{cccccc}
1 & 2 & 2 & 3 & \leftarrow & 2 \\
2 & 3 & 5 & 5 & & \\
4 & 4 & 6 & & & \\
5 & 6 & & & &
\end{array}
\quad \Rightarrow \quad
\begin{array}{cccccc}
1 & 2 & 2 & 2 & & \\
2 & 3 & 5 & 5 & \leftarrow & 3 \\
4 & 4 & 6 & & & \\
5 & 6 & & & &
\end{array}
$$

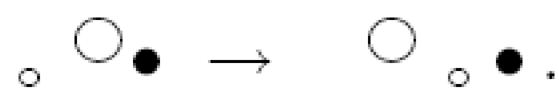
$$
\Rightarrow \quad
\begin{array}{cccccc}
1 & 2 & 2 & 2 & & \\
2 & 3 & 3 & 5 & & \\
4 & 4 & 6 & & \leftarrow & 5 \\
5 & 6 & & & &
\end{array}
\quad \Rightarrow \quad
\begin{array}{cccc}
1 & 2 & 2 & 2 \\
2 & 3 & 3 & 5 \\
4 & 4 & 5 & \\
5 & 6 & 6 &
\end{array}
\quad .
$$

Knuth moves.

A Knuth move of type A on a three letter word yzx is to replace yzx by yxz if $x < y \leq z$. Symbolically, we can remember this as

$A:$ 

A Knuth move of type B on a three letter word sends $xzy \rightarrow zxy$ if $x \leq y < z$. Symbolically we can remember this as

$B:$ 

We can achieve the bumping algorithm more slowly if we place the letter n to the right of the word of T and then successively apply moves of type A and B, whichever is appropriate, to the three letter subwords starting at the right.

Example

$$
\begin{aligned}
56446235512232 &\xrightarrow{B} 56446235512322\\
&\xrightarrow{B} 56446235513222\\
&\xrightarrow{B} 56446235531222\\
&\xrightarrow{A} 56446235351222\\
&\xrightarrow{B} 56446253351222\\
&\xrightarrow{B} 56446523351222\\
&\xrightarrow{B} 56464523351222\\
&\xrightarrow{B} 56644523351222.
\end{aligned}
$$

$$A: \quad \bullet^{\bigcirc}{}_{\circ} \to \bullet{}_{\circ}{}^{\bigcirc}.$$

replace yzx by yxz if $x < y \le z$.

$$B: \quad {}_{\circ}{}^{\bigcirc}\bullet \quad \to \quad {}^{\bigcirc}{}_{\circ}\bullet .$$

$xzy \to zxy$ if $x \le y < z$.

$$
\begin{array}{cccc}
1 & 2 & 2 & 3\\
2 & 3 & 5 & 5\\
4 & 4 & 6 & \\
5 & 6 & &
\end{array}
\quad \Rightarrow \quad
\begin{array}{cccc}
1 & 2 & 2 & 2\\
2 & 3 & 3 & 5\\
4 & 4 & 5 & \\
5 & 6 & 6 &
\end{array}
\quad .
$$

Why Knuth moves give slow bumping.

$A: \quad \bullet^{\bigcirc}{}_{\circ} \to \bullet{}_{\circ}{}^{\bigcirc}.$

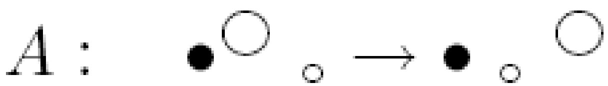
replace yzx by yxz if $x < y \leq z$.

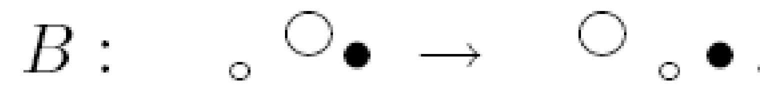
$B: \quad {}_{\circ}{}^{\bigcirc}\bullet \to {}^{\bigcirc}{}_{\circ}\bullet.$

$xzy \to zxy$ if $x \leq y < z$.

To see that this slow bumping always does achieve the bumping consider the top row (with x adjoined) in the form $u_1 \dots u_p x' v_1 \dots v_q x$ where $x' > x$ and $u_p \leq x$. Then $v_q \geq v_{q-1} > x$ and so the Knuth operation A moves x to the left of v_q and this continues (with A) until the configuration $u_1 \dots u_p x' x v_1 \dots v_q$ is reached. Then $u_p \leq x$ and $x' > x$ so operation B moves x' to the left of u_p and we can apply B to the configuration $u_{p-1} x' u_p$ to move the x' to the left of u_{p-1}. This continues until x' has been moved to the left of the entire row, and so has been bumped to the second row, etc.

Elementary Knuth moves





• = y

An **elementary Knuth transformation** on a word applies one of A or B or their inverses to three successive letters of a word. So an elementary Knuth equivalence changes the order of two letters on one side of a letter y if one is bigger and the other is smaller than y.

Knuth equivalence.

Two words are **Knuth equivalent** if they can be obtained from one another by a succession of elementary Knuth transformations. What we have shown is that

Proposition 4.1 *If $w(T)$ is the word of the tableau T, then the word $w(T) \cdot x$ is Knuth equivalent to the word of the tableau obtained from T by inserting the letter x and using the Shensted bumping algorithm.*

Words and tableaux.

A word consisting of one letter is obviously the word of the tableau with one box. If $w = x_1 \cdots x_p$ is any word, then the proposition (applied inductively) implies that w is Knuth equivalent to the word of a tableau. We have thus proved:

Proposition 4.2 *Any word is Knuth equivalent to the word of a tableau.*

What is hard to prove is

Theorem 4.1 *Every word is Knuth equivalent to the word of a unique tableau.*

Web site.

4.1 Web site for listing of all words equivalent to a given word,

http://www.mathe2.uni-bayreuth.de/axel/all_plactic_word.html

You type in the word in the form [a,b,c,d,..] and then hit [berechnen] and you will get the list of all words Knuth equivalent to the word you typed.

Increasing sequences in a word.

If $w = x_1 x_2 \ldots x_r$ is a word, an **increasing sequence** of w is a sequence of integers $i_1 < \cdots < i_p$ such that $x_{i_1} \leq \cdots \leq x_{i_p}$ and then p is called the length of the sequence. For example, if

$$w = 134234122332$$

then

$$\underline{1}34\underline{2}341\underline{2233}2$$

extracts an increasing sequence of length six as does

$$\underline{1}34234\underline{12233}2.$$

There are no increasing sequences of longer length. We write $L(w, 1) = 6$. So in general $L(w, 1)$ denotes the longest length of an increasing sequence in the word.

L(w,k).

For any integer k, let $L(w,k)$ denote the largest integer which can be realized as the sum of the lengths of k disjoint increasing sequences. For example, the above word has $L(w,2) = 9$ because we can extract the two increasing sequences

$$\underline{1}\overline{34}\underline{2}3\overline{4}1\underline{2233}2$$

of lengths six and three and can not do any better. We also have $L(w,3) = 12$ because we have

$$\underline{1}\overline{34}\underline{23}\overline{4}\mathbf{122}\underline{33}\mathbf{2}$$

disjoint increasing sequences of lengths 5,4, and 3. We can not do better, and there is no collection of disjoint increasing sequences of lengths 6,3, and 3, for example. Since there are 12 letters in the word, we must have $L(w,k) = 12$ for all $k \geq 3$.

The word constructed by the bumping algorithm.

$$w = 134234122332$$

$$1 \to 13 \to 134 \to \begin{array}{ccc} 1 & 2 & 4 \\ 3 & & \end{array} \to \begin{array}{ccc} 1 & 2 & 3 \\ 3 & 4 & \end{array} \to \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 4 & & \end{array} \to \begin{array}{cccc} 1 & 1 & 3 & 4 \\ 2 & 4 & & \\ 3 & & & \end{array}$$

$$\to \begin{array}{cccc} 1 & 1 & 2 & 4 \\ 2 & 3 & & \\ 3 & 4 & & \end{array} \to \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & 3 & 4 & \\ 3 & 4 & & \end{array} \to \begin{array}{ccccc} 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 4 & & \\ 3 & 4 & & & \end{array} \to \begin{array}{cccccc} 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 3 & 4 & & & \\ 3 & 4 & & & & \end{array}$$

$$\to \begin{array}{cccccc} 1 & 1 & 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & & & \\ 3 & 4 & 4 & & & \end{array} .$$

The word of this tableau is 344233112223 and we can conveniently read off the increasing sequences from the tableau starting at the top.

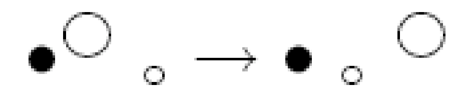
In fact it is clear that if z is the word of a tableau of shape λ then $L(z,k)$ is the sum of the number of boxes in the first k rows of λ. Suppose that we could prove that $w \equiv w'$ implies that $L(w,k) = L(w',k)$. Then if $w \equiv w'$ are both words of a tableau, we would know at least that the shape of these tableaux are the same.

So as a first step we want to prove that

$$w \equiv w' \textbf{ implies that } L(w,k) = L(w',k).$$

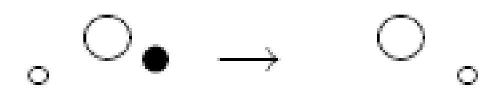
Proof. We need only check this for the elementary Knuth transformations. So let w and w' be the left and right hand sides of

$$u \cdot yxz \cdot v \equiv u \cdot yzx \cdot v \quad x < y \leq z \qquad A:$$



and

$$u \cdot xzy \cdot v \equiv u \cdot zxy \cdot v \quad x \leq y < z. \quad B:$$



Any disjoint collection of k increasing sequences for w' is a disjoint collection of k increasing sequences for w. So $L(w,k) \geq L(w',k)$. We must prove the reverse inequality. Any increasing sequence for w will be an increasing sequence for w' unless it contains both x and z. So suppose that one of the sequences for w is of the form $a \cdot xz \cdot b$. If no other sequence we are using contains y, then $a \cdot yz \cdot b$ is an increasing sequence for w' in the first case

and $a \cdot xy \cdot b$ is an increasing sequence in the second case

So we get the same sum of lengths.

w w'

$$u \cdot yxz \cdot v \equiv u \cdot yzx \cdot v \quad x < y \leq z$$

$$u \cdot xzy \cdot v \equiv u \cdot zxy \cdot v \quad x \leq y < z.$$

one of the sequences for w is of the form $a \cdot xz \cdot b$.

some other sequence we are using for w is of the

form $c \cdot y \cdot d$. In the first case replace the two sequences by $c \cdot yz \cdot b$ and $a \cdot x \cdot d$. In the second case replace the two sequences used in w by $a \cdot xy \cdot d$ and $c \cdot x \cdot b$. QED

Removing the largest letter.

We now want to complete the proof that each Knuth equivalence class contains exactly one word of a tableau. We already know that the shape of this word is determined. In a tableau T for which n is the largest integer, an n must occur at a **corner**, i.e. at a position which is at the right hand end of its row and the bottom of its column. In fact, suppose that n is the largest integer in the word w, and we let $w_\circ$ denote the word obtained from w by deleting the rightmost n which occurs in w. Let λ be the shape of the tableau associated with w, and let $\lambda_\circ$ be the shape of the tableau associated to $w_\circ$. If $w = w(T)$ is the word of a tableau, then $w_\circ = w(T_\circ)$ is the word of the tableau $T_\circ$ obtained from T by removing n from the (end of the) highest row in which it appears. Suppose we could prove the following lemma:

Lemma 5.2 *If w and w' are any words, then $w \equiv w' \Rightarrow w_\circ \equiv w'_\circ$.*

Proof of theorem assuming the lemma. We can now prove the theorem, which asserts that every Knuth equivalence class contains a unique word $w(T)$ that is the word of a tableau by induction on the length, i.e number of letters in the word. Clearly every word of length one is the word of a tableau and no two distinct words of length one are equivalent. So the theorem is true for words of length one. Suppose we know the theorem for all words of length $\leq N-1$. If $w = w(T)$ is of length N, and n is the largest letter occurring in w, then the shape $\lambda = \lambda(T)$ is determined, and so is $\lambda_\circ = \lambda(T_\circ) = \lambda(w_\circ)$ which is obtained from λ by removal of a corner. So we know that this corner in λ must contain an n. If $w \equiv w'$ and w' is the word of a tableau, then since $w_\circ \equiv w'_\circ$ the shapes of the tableaux associated to w' and $w'_\circ$ are the same as those associated to w and $w_\circ$ and this same corner in T and T' must contain an n. But now $T'_\circ$ must be identical with $T_\circ$ since $w_\circ \equiv w'_\circ$ and these words have length $N-1$. This implies that $T = T'$.

Proof of the lemma.

Lemma 5.2 *If w and w' are any words, then $w \equiv w' \Rightarrow w_\circ \equiv w'_\circ$.*

Proof of the lemma. To prove this it is enough to prove this for the elementary equivalences

$$u \cdot yxz \cdot v \equiv u \cdot yzx \cdot v \quad x < y \leq z$$

and

$$u \cdot xzy \cdot v \equiv u \cdot zxy \cdot v \quad x \leq y < z.$$

If the n that is removed is not one of the xyz the result follows by induction. If the n that is removed *is* one of the xyz then it must be the z, and then the resulting words are the same. QED

The plactic monoid.

An associative monoid is a set with a binary operation (called multiplication) for which the associative law holds. For example, the collection of all words on a given alphabet is a monoid in which multiplication is juxtaposition. It is called the free associative monoid on the letters of the given alphabet. If the letters are $\{1, \ldots, m\}$ this monoid is denoted by $F[m]$ or simply by F if m is understood. the empty word $\emptyset$ is a left and right identity on F. If two words are Knuth equivalent, so is their product. Hence the multiplication on F descends to a multiplication on the set M of Knuth equivalence classes of words. This makes M into an associative monoid with left and right identity, which is called the **plactic monoid**.

The bumping algorithm for multiplication.

Each element of M corresponds to a unique tableau by Theorem 4.1. So the product of two elements of M can be thought of as a multiplication on tableaux. By the associative law, the product $T \cdot U$ of two tableau can be computed as follows: Start with the tableau T, and insert the leftmost entry of the bottom row of U into T. Then insert the next entry, etc. In other words, successively insert the letters of $w(U)$ into T reading from left to right. For example.

$$\begin{array}{cccc} 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 5 \\ 4 & 4 & 6 & \\ 5 & 6 & & \end{array} \bullet \begin{array}{cc} 1 & 3 \\ 2 & \end{array} = \begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 3 & 3 & 5 \\ 4 & 4 & 5 & \\ 5 & 6 & 6 & \end{array} \bullet 1 \; 3 = \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 5 \\ 3 & 4 & 5 & \\ 4 & 6 & 6 & \\ 5 & & & \end{array} \bullet 3$$

$$= \begin{array}{ccccc} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 5 & \\ 3 & 4 & 5 & & \\ 4 & 6 & 6 & & \\ 5 & & & & \end{array} .$$

The bumping algorithm for multiplication.

$$\begin{matrix} 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 5 \\ 4 & 4 & 6 & \\ 5 & 6 & & \end{matrix} \bullet \begin{matrix} 1 & 3 \\ 2 & \end{matrix} = \begin{matrix} 1 & 2 & 2 & 2 \\ 2 & 3 & 3 & 5 \\ 4 & 4 & 5 & \\ 5 & 6 & 6 & \end{matrix} \bullet 1\ 3 = \begin{matrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 5 \\ 3 & 4 & 5 & \\ 4 & 6 & 6 & \\ 5 & & & \end{matrix} \bullet 3$$

$$= \begin{matrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 5 & \\ 3 & 4 & 5 & & \\ 4 & 6 & 6 & & \\ 5 & & & & \end{matrix} .$$

If ν is the shape of this product tableau, then ν contains λ, the shape of T as a subdiagram. This is obviously a general fact.

Multiplication by a row

We want to examine some properties of this multiplication in the special case where U is row or where U is a column. Consider the multiplication

$$\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 3 & 3 & 5 \\ 4 & 4 & 5 & \\ 5 & 6 & 6 & \end{array} \quad \bullet 1\ 3 = \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 5 \\ 3 & 4 & 5 & \\ 4 & 6 & 6 & \\ 5 & & & \end{array} \quad \bullet\, 3 = \begin{array}{ccccc} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 5 & \\ 3 & 4 & 5 & & \\ 4 & 6 & 6 & & \\ 5 & & & & \end{array}$$

Let ν be the shape of this product tableau, and λ, the shape of T Let ν/λ denote the **skew diagram** obtained by removing the boxes of λ from μ. Notice that no two boxes in ν/λ lie in the same column. (There is a box of ν/λ in the first and fifth column.) This is also a general fact:

Multiplication by a row, 2.

Proposition 6.1 *If λ is the shape of T and ν is the shape of $T \bullet U$ then no two boxes in the skew diagram ν/λ lie in the same column.*

In order to prove this, it is convenient to introduce the notion of the **bumping route** of an insertion. Inserting the number x into a tableau T determines a collection R of boxes consisting of the positions where an element was bumped, together with the box where the last bumped element lands up. In our example at the beginning, the bumping route consists of the boxes containing bold face numbers below:

$$\begin{matrix} 1 & 2 & 2 & \mathbf{2} \\ 2 & 3 & \mathbf{3} & 5 \\ 4 & 4 & \mathbf{5} & \\ 5 & 6 & \mathbf{6} & \end{matrix}$$

The bumping route.

$$
\begin{array}{cccc}
1 & 2 & 2 & 3 \\
2 & 3 & 5 & 5 \\
4 & 4 & 6 & \\
5 & 6 & &
\end{array}
\leftarrow 2
\quad \Rightarrow \quad
\begin{array}{cccc}
1 & 2 & 2 & 2 \\
2 & 3 & 5 & 5 \\
4 & 4 & 6 & \\
5 & 6 & &
\end{array}
\begin{array}{c}
\\
\leftarrow 3 \\
\\
\\
\end{array}
$$

$$
\Rightarrow \quad
\begin{array}{cccc}
1 & 2 & 2 & 2 \\
2 & 3 & 3 & 5 \\
4 & 4 & 6 & \\
5 & 6 & &
\end{array}
\begin{array}{c}
\\
\\
\leftarrow 5 \\
\\
\end{array}
\quad \Rightarrow \quad
\begin{array}{cccc}
1 & 2 & 2 & 2 \\
2 & 3 & 3 & 5 \\
4 & 4 & 5 & \\
5 & 6 & 6 &
\end{array}
.
$$

$$
\begin{array}{cccc}
1 & 2 & 2 & \mathbf{2} \\
2 & 3 & \mathbf{3} & 5 \\
4 & 4 & \mathbf{5} & \\
5 & 6 & \mathbf{6} &
\end{array}
$$

The bumping routes of multiplication by a row.

Consider the multiplication

$$\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 3 & 3 & 5 \\ 4 & 4 & 5 & \\ 5 & 6 & 6 & \end{array} \bullet 1\ 3 = \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 5 \\ 3 & 4 & 5 & \\ 4 & 6 & 6 & \\ 5 & & & \end{array} \bullet 3 = \begin{array}{ccccc} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 5 & \\ 3 & 4 & 5 & & \\ 4 & 6 & 6 & & \\ 5 & & & & \end{array} .$$

If we draw the bumping route of the first insertion in the above multiplication we get

$$\begin{array}{cccc} 1 & \mathbf{1} & 2 & 2 \\ 2 & \mathbf{2} & 3 & 5 \\ \mathbf{3} & 4 & 5 & \\ \mathbf{4} & 6 & 6 & \\ \mathbf{5} & & & \end{array}$$

which clearly lies to the left of the next insertion which consists of simply adding a 3 to the first row.

Lemma 6.1 *Insert x into the tableau T and then insert x' into the resulting tableau.*

- *If $x \le x'$ then the bumping route R of the x-insertion lies strictly to the left of the bumping route R' of the second insertion, and the new box B of the first insertion lies strictly left and weakly below the new box B' of the second insertion.*

- *If $x > x'$ then R' is weakly left of R and B' is weakly left and strictly below B.*

Proof. Suppose $x \le x'$. If x causes no bumping on the top row, then neither does x' which is placed immediately to the right of x in the final diagram. If x bumps a number y from the first row, then x' does not bump x from the first row since $x \le x'$. So if x' bumps anything from the first row, it must bump a number y' from a box strictly to the right of the position of the y bumped by x, and so $y \le y'$.

Proof. Suppose $x \leq x'$. If x causes no bumping on the top row, then neither does x' which is placed immediately to the right of x in the final diagram. If x bumps a number y from the first row, then x' does not bump x from the first row since $x \leq x'$. So if x' bumps anything from the first row, it must bump a number y' from a box strictly to the right of the position of the y bumped by x, and so $y \leq y'$. Iterating this argument shows that R lies strictly to the left of R'. Also, at each stage the insertion of a z into a row (for the first insertion) is followed by and insertion of a $z' \geq z$ (or no insertion at all) into this row. So R can not terminate in a row strictly above the termination of R'. If R' stops first then R keeps moving down and weakly to the left so B is strictly to the left and weakly below B'.

Suppose that $x > x'$. If x does not bump an element from the top row then x' certainly does, and the bumped element lies weakly to the left of the position occupied by x, so R' lies weakly to the left and strictly below R. If x does bump an element y from the first row, then $y > x$ and the element y' bumped by x' is $\leq x$ so R' is weakly to the left of R on the top row and the process continues. This completes the proof of the lemma. The proposition follows from the first case of the lemma, since each new box inserted lies strictly to the right of the preceding one.

Proposition 6.1 *If λ is the shape of T and ν is the shape of $T \bullet U$ then no two boxes in the skew diagram ν/λ lie in the same column.*

The proposition has a converse. Before stating it, we observe that the Shensted bumping algorithm has an inverse in the following sense. Suppose we start with a tableau and with a given corner of its diagram, and are told that we obtained this entry by a bumping. For example, suppose we are given the tableau

$$\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 3 & 3 & 5 \\ 4 & 4 & 5 & \\ 5 & 6 & 6 & \end{array}$$

together with the information that the "new" box is the corner occupied by the 6. Then we can retrace the bumping route by doing reverse bumping: The 6 bumps the 5 in the third row which bumps the rightmost 3 in the second row which bumps rightmost 2 in the first row out of the diagram.

In general, if z denotes the entry in the

"new" box, it looks for the entry in the row above which is strictly less than z and which is furthest to the right, and bumps this entry out; the process continues until an element is bumped out of the first row.

This shows that if we are given a tableau X with shape ν, a subdiagram λ such that ν/λ has all its boxes in different columns, then we can find a unique tableau T and a unique tableau U whose shape is a row such that $X = T \bullet U$.

Multilplication by a column.

We can use the second half of the lemma to characterize the shape of multiplication by a column. Again we use the notation λ for the shape of T and ν for the shape of $T \bullet U$:

Proposition 6.2 *If the shape of* U *is a column then no two boxes of* ν/λ *lie in the same row. Conversely, if* X *is a tableau of shape* ν *and* λ *is a subdiagram of* ν *such that no two boxes of* ν/λ *lie in the same row, then there is a tableau* T *os shape* λ *and a tableau* U *whose shape is a column such that* $X = T \bullet U$.

The plactic monoid.

Given any monoid we can form an algebra by considering the set of linear combinations of the elements of the monoid with coefficients in a fixed ring (say the integers or the complex numbers). Multiplication of two basis elements is determined by the monoid structure which then extends by bilinearity to all linear combinations. We will denote the algebra formed from M by R (or by $R_{[m]}$ when we need to specify the alphabet, or by $R_{[m]}(\mathbb{Z})$ if we also need to specify that the ring of coefficients is $\mathbb{Z}$).

The Pieri formulas.

For a Young diagram λ, we let $S_\lambda \in R$ denote the sum over all tableaux of shape λ. Of great interest will be a formula for the product in R of two elements of this type, i.e for the product $S_\lambda \bullet S_\mu$. This will require some work. But the two preceding propositions allow us to compute this product when μ is a row or a column:

Let (p) denote the Young diagram with one row and p columns. Then

$$S_\lambda \bullet S_{(p)} = \sum_\nu S_\nu \tag{1}$$

where the sum is over all ν which can be obtained from λ by adding p boxes with no two boxes in the same column. Indeed, given a tableau of shape λ and a row U then we get a term occurring exactly once in one of the summands on the right, and given any tableau X occurring a summand in one of the S_ν in the sum on the right, it can be written uniquely as $X = T \bullet U$ where T is a tableau of shape λ and U is a tableau whose shape is (p).

If (1^p) denotes the tableau consisting of one column and p rows, then a similar argument (using Proposition 6.2 shows that

$$S_\lambda \bullet S_{(1^p)} = \sum_\nu S_\nu \tag{2}$$

where the sum is over all ν which can be obtained from λ by adding p boxes with no two boxes in the same row.