MATH 126
SOLUTION SET 3

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1. $\chi(g) = \mathrm{tr}(\rho(g)) = \sum_i \lambda_i$, where λ_i are the characteristic values of $\rho(g)$. Now $|\sum_i \lambda_i| \le \sum_i |\lambda_i|$, with equality iff all the λ are equal. Since all the λ_i are roots of unity[1], $|\lambda_i| = 1$, so $|\sum_i \lambda_i| \le \sum 1 = d$, with equality iff $\lambda_i = \lambda$, which is the case iff $\rho(g) = \lambda I$.
2. You should *not* use contradiction here; it is straight-forward. $\hat{\rho}(G) = \rho(G/H)$ (because π is surjective), and since there are no subspaces invariant under the latter, there are none under the former.
3. **Associative:** The tensor product is associative; you can take this as known, but should mention it[2]. Also, for characters the tensor product corresponds to the standard product on $\mathbf{C}$, which is associative.

   Identity: The trivial character is the identity:

$$(1 \otimes \chi)(g) = 1\chi(g) = \chi(g) \tag{3.1}$$

   Composition: The tensor product of two representations is a representation, and if the two representations are one-dimensional, their product will be too, since $1 \cdot 1 = 1$.

   Inverse: Since $\mathrm{GL}(\mathbf{C}) = \mathrm{C}^*$, $\chi^{-1}(g) = \chi(g)^{-1}$ is well-defined[3], and

$$\chi^{-1}(gh) = \chi(gh)^{-1} = (\chi(g)\chi(h))^{-1} = \chi(h)^{-1}\chi(g)^{-1} = \chi(g)^{-1}\chi(h)^{-1} \tag{3.2}$$

   since $\mathrm{GL}(\mathbf{C})$ is commutative, so this defines a one-dimensional character such that $\chi^{-1}(g) \otimes \chi(g) = 1$, so it is the inverse of χ.

   Note that since $\mathbf{C}$ is commutative, this group is in fact commutative.
4. This solution was given (in different forms) by several students; I have rewritten it a bit.

   $\mathbf{C}[G]$ can be view as a $\mathbf{C}[G]$ module in more than one way: if we define $g(h) = gh$, we get the regular representation. However, if we define $g(h) = ghg^{-1}$ we get another representation[4], which we shall call the conjugation representation. This is a kind of permutation representation, so its character is the number of fixed points, so $\pi(g) = \frac{|G|}{|C_g|}$ (by orbit-stabilizer), where C_g denotes the conjugacy class of g and π is the character of our representation. Now given an irreducible character (indeed, any character) χ, $\langle \pi, \chi \rangle$ is a non-negative integer. Also,

$$\langle \pi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \frac{|G|}{|C_g|} \chi(g) = \sum_{g \in G} \frac{1}{|C_g|} \chi(g) \tag{3.3}$$

   which is just the sum of χ's row in the character table. Thus, this sum is a non-negative integer, as desired.
5. Again, culled from students.

   (A) Every index two subgroup is normal[5], thus we can quotient out by H and get $G/H = C_2 = \{1, -1\}$. By question 4, we can sum across conjugacy classes and get a non-negative integer. However,

$$\chi([g]) = \begin{cases} 1 & [g] \subset H \\ -1 & [g] \not\subset H \end{cases} \tag{3.4}$$

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[1] Since $\rho(g)$ has finite order, since G is finite, remember?

[2] This follows immediately from category theory; both $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ are trilinear, and since the tensor product is the solution to a universal mapping property, it is unique up to a unique isomorphism. Thus, $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ by a unique isomorphism.

[3] Since one dimensional characters are never zero.

[4] This is easily verified to be a representation.

[5] Because $G = H \coprod sH = H \coprod Hs$, so $sH = Hs$, so H is normal.

so this sum is the number of conjugacy classes in H minus the number of conjugacy classes not in H. Thus, there are at least as many of the former as the latter.

In particular, $[S_n : A_n] = 2$, so this holds for A_n (even permutations) having more classes than $S_n \setminus A_n$ (odd permutations).

(B) Again, quotient out by H to get $\pi\colon G \to G/H = C_3 = \{1, \zeta_3, \zeta_3^2\}$. So $H = \pi^{-1}(1)$ is a union of conjugacy classes (the conjugacy classes which get mapped to 1), and similarly for $sH = \pi^{-1}(\zeta_3)$, $tH = \pi^{-1}(\zeta_3^2)$. Summing across conjugacy classes as in question 4, we get a non-negative integer. In particular, this is real, so sH, tH have the same number of conjugacy classes (since we need the ζ, ζ^2 to cancel). Also, it is non-negative, and since $\zeta_3 + \zeta_3^2 = -1$, we see that H has at least as many conjugacy classes as sH, tH.

6. I did parts A, B, D, E, F, H and students did C, G (though I polished the solutions).

	1	C	ι	Δ	∇	H	H'
$\lvert C_g \rvert$	1	6	7	7	7	7	7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	-1	ω^4	ω^2	ω^1	ω^5
χ_3	1	1	1	ω^2	ω^4	ω^2	ω^4
χ_4	1	1	-1	1	1	ω^3	ω^3
χ_5	1	1	1	ω^4	ω^2	ω^4	ω^2
χ_6	1	1	-1	ω^2	ω^4	ω^5	ω^1
χ_7	6	-1	0	0	0	0	0

Character Table 1: G

(A) This character table is obtained as follows:
- $\chi_2(G)$ forms a group which contains ω, so it must contain ω^5. Thus, $\chi_2(H) = \omega^5$
- Tensoring χ_2 with itself yields more representations, which are $\chi_3, \dots, \chi_6$.[6]
- χ_7 must have zeros except for $1, C$, since otherwise tensoring with χ_2 would yield another simple character. It remains to determine $\chi_7(1), \chi_7(C)$. Because χ_2 gives a quotient $G \to C_6$, we know that $6 \mid |G|$. Also, $\chi_7(1)^2 + 6 = |G|$, so $6 \mid \chi_7(1)$. However, we know that the dimension of a representation divides the order of the group, so $\chi_7(1) \mid \chi_7(1)^2 + 6$, so $\chi_7(1) \mid 6$. Since $\chi_7(1)$ is a positive integer, $\chi_7(1) = 6$. Lastly, by column orthogonality of the first and second columns, $6 + \chi_7(1)\chi_7(C) = 0$, so $\chi_7(1)\chi_7(C) = -6$, so $\chi_7(C) = -1$.

(B) The number of elements in each conjugacy class can be determined by multiplying the character table by its conjugate transpose[7] and looking at the diagonal entries, which give us the order of the stabilizers of elements in each class. These are included in the character table above.

(C) Clearly, 1 has order 1. From part D we see that the elements of C have order 7. Elements of H, H' must have order divided by 6, so order $6, 42$. The latter is impossible, as G isn't cyclic, so they have order 6.

The orders of the rest follow from the determination of Sylow subgroups in part G. To summarize:

Conjugacy Class	1	C	ι	Δ	∇	H	H'
Order of Elements	1	7	2	3	3	6	6

Table 1. Order of Elements

(D) The commutator subgroup is the kernel of the one-dimensional characters, so it's $1 \coprod C = C_7$, since this is the only group of order 7.

(E) χ_3 gives us a map $G \twoheadrightarrow C_3$, so the kernel of this map has order $42/3 = 14$. We'll denote this group by $\mathfrak{H}$ to avoid confusion with the conjugacy class H.

[6]We use ω^3 instead of -1 to make the group structure visible.

[7]Also a good way to check your work.

(F) $\mathfrak{H}$ is a non-commutative (since we haven't quotiented out by the commutator) group of order $14 = 2 \cdot 7$, so $\mathfrak{H} = D_7$.

(G) By the Sylow theorems, we know that we have subgroups of order $2, 3, 7$. Let s_p be the number of Sylow subgroups of order p. By the Third Sylow Theorem: $s_p | |G| / p = 42/p$, since $|G| = 2 \cdot 3 \cdot 7$; and $s_p \equiv 1 \mod p$. From these we immediately obtain s_3, s_7. $s_7 = 1$ so there is 1 7-Sylow group (namely $1 \coprod C$). $s_3 = 7$ (these correspond to elements in Δ and their square in ∇, since elements of C, ι, H, H' cannot have the correct order). Lastly, we must have some 2-Sylow group, and the only elements with order 2 possible are the remaining ones, namely ι. These are all conjugate, so there are 7 2-Sylow groups.

To summarize (and then a bit):

Subgroup	1	$1 \coprod C$	$\{1, i\}, i \in \iota$	$\{1, \delta, \delta^2\}$	$\{1, h, \dots, h^5\}$	$\{1, h', \dots, (h')^5\}$
Number of Conjugate	1	1	7	7	7	7

TABLE 2. Sylow and other cyclic subgroups, in conjugate groups

where $\delta \in \Delta, \delta^2 \in \nabla, h \in H, h' \in H'$

(H) The restrictions of one-dimensional representations for G are irreducible representations for H (either χ_1' or χ_2', depending on where they send ι), so it suffices to decompose χ_7. The character table for D_7 is (from a previous homework):

	1	$\{c^1, c^6\}$	$\{c^2, c^5\}$	$\{c^3, c^4\}$	ι
χ_1'	1	1	1	1	-1
χ_2'	1	1	1	1	1
χ_3'	2	$\zeta^1 + \zeta^6$	$\zeta^2 + \zeta^5$	$\zeta^3 + \zeta^4$	0
χ_4'	2	$\zeta^2 + \zeta^5$	$\zeta^3 + \zeta^4$	$\zeta^1 + \zeta^6$	0
χ_5'	2	$\zeta^3 + \zeta^4$	$\zeta^1 + \zeta^6$	$\zeta^2 + \zeta^5$	0
$\mathrm{Res}_H^G(\chi_7)$	6	-1	-1	-1	0

CHARACTER TABLE 2: D_7 (with an extra character)

where ζ is a primitive seventh root of unity, and we are considering $D_7 \lhd G, D_7 = 1 \coprod C \coprod \iota$. You *could* compute inner products of these to conclude that $\mathrm{Res}_H^G(\chi_7) = \chi_3' \oplus \chi_4' \oplus \chi_5'$, or you could observe[8] that this works, and since simple characters form a basis for class functions, this is the unique solution.

[8]You should notice this, since the n-th roots of unity sum to zero. Plus, you should conjecture that it would be symmetric.