TRANSLATING THE IRREDUCIBLE REPRESENTATIONS OF S_n INTO $GL_n(\mathbf{F}_q)$

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1. Introduction

1.1. **The Argument.** In both S_n and $GL_n(\mathbf{F}_q)$, we send letters (respectively, basis elements) to other letters (basis elements). In GL_n we can take linear combinations, which we can't do in S_n; but this is really the only difference. GL_n, loosely speaking, is an obese, redundant version of the svelte S_n.

The goal of this paper is to make this correspondence precise, to show that, under this correspondence between letters and bases, we can import many of GL_n's representations wholesale from S_n.

Section 2 is based on [2]. Section 3 is based on [1].

1.2. **Permutation Representations in S_n.** To begin, we clarify the representations of S_n. All S_n's representations begin with the permutation representations; though these aren't themselves irreducible, we can construct all the irreducibles from them. We saw in class the simplest kind of permutation representation, the representation of how a permutation acts on single letters. We can look at a more general kind of representation, one acting on a group of letters simultaneously.

More precisely, for each partition μ of n, we define tableaux: write a table with μ_1 columns in the first row, μ_2 columns in the second row, and so on. In each cell, put one number $1, 2, \ldots, n$, each appearing exactly once in the diagram.

Here is a tableau of type $(1, 2)$: $\begin{smallmatrix}1 & \\ 2 & 3\end{smallmatrix}$

Here is different tableau of type $(1, 2)$: $\begin{smallmatrix}1 & \\ 3 & 2\end{smallmatrix}$

Here is a tableau of type $(1, 4, 5)$: $\begin{smallmatrix}5 & & & & \\ 2 & 1 & 9 & 10 & \\ 3 & 4 & 6 & 7 & 8\end{smallmatrix}$

This isn't quite the concept we want to use for defining the permutation representations of S_n. Instead, let's define a tabloid as a tableau modded out by rearranging the letters in a row. So, as tabloids, $\overline{\begin{smallmatrix}1 & \\ 2 & 3\end{smallmatrix}}$ is the same as $\overline{\begin{smallmatrix}1 & \\ 3 & 2\end{smallmatrix}}$, but different from $\overline{\begin{smallmatrix}2 & \\ 1 & 3\end{smallmatrix}}$.

We put a bar over tabloids to show that you can rearrange the letters in any row. As notation, we put $\{t\}$ to be t's associated tabloid.

Now, for each partition μ, S_n has a permutation representation on tabloids of type μ, with the natural action.

2. Some irreducibles of GL_n

The corresponding concept in GL_n is flags. Given a partition $\mu = 0 \leq \mu_1 \leq \mu_2 \leq \mu_n$ of n, we define a flag $e(\mu)$ of type μ as a collection of subspaces $V_1 \subset V_2 \subset \cdots \subset V_m$, where $\dim V_i = \mu_1 + \cdots + \mu_i$. That is, $e(\mu)$ is some flag of the same shape as our partition.

Since all $g \in GL_n$ are bijections, $\dim g(V_i) = \dim V_i$; i.e., g permutes flags of type μ. So we can consider the permutation character $\rho(\mu)$ for flags of type μ.

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Corresponding to this is $R(\mu)$, the permutation character of S_n on tabloids of type μ.

Before we compute our irreducibles, we need to prove two linear-algebra lemmata.

Lemma 2.1. *Let e and f be two flags of type $\overbrace{(1,1,\ldots,1)}^{n} = 1^n$. Then we can choose vectors $v_1,\ldots,v_n$, such that $\langle v_1,\ldots,v_i\rangle = e_i$ and $\langle v_{\pi(1)},\ldots,v_{\pi(i)}\rangle = f_i$ for some permutation π.*

Proof. By induction on n. If $n = 1$, this is trivial: $e = f$.

If $n > 1$, proceed as follows: Note that $e' = e \cap e_{n-1}$ and $f' = f \cap e_{n-1}$ are both flags of type 1^{n-1}. (By $f \cap e_{n-1}$, we mean the flag $f_1 \cap e_{n-1} \subset f_2 \cap e_{n-1}$. This is a flag of type 1^{n-1} since f ends at $f_n \cap e_{n-1} = V \cap e_{n-1} = e_{n-1}$, which has dimension $n-1$, and at each step $f_i \cap e_{n-1}$ can only increase in dimension by one, since f_i does so.)

So, by induction, we can choose vectors $v_1,\ldots v_{n-1}$ so that $\langle v_1,\ldots,v_i\rangle = e'_i$ and $\langle v_{\pi(1)},\ldots,v_{\pi(i)}\rangle = f'_i$.

Choose z so that $f_z \subset e_{n-1}$ but $f_{z+1} \not\subset e_{n-1}$. (There must be some such z, since $f(0) = \emptyset \subset e_{n-1}$ but $f(n) = V \not\subset e_{n-1}$.) Now, choose v_n to be any element of $f_{z+1} \setminus f_z$.

First, $\langle v_1,\ldots,v_i\rangle = e_i$:

For $i < n$, there is no problem: $\langle v_1,\ldots,v_i\rangle = e'_i = e_i$.

Note, since $v_n \notin f_i$, $e'_{z+1} = f_z \oplus v_n$. Since $f_z \subset e_{n-1}$ but $f_{z+1} \not\subset e_{n-1}$, it must be that $v_n \notin e_{n-1}$. So $\langle v_1,\ldots,v_n\rangle = e_{n-1} \oplus v_n = V$, as desired.

Second, with the order

$$v_1, v_2, \ldots, v_{\pi(z)}, v_n, v_{\pi(z+1)}, \ldots v_{\pi(n-1)},$$

the v_i span f too:

For $i \le z$, $\langle v_{\pi(1)},\ldots,v_{\pi(i)}\rangle = f'_i = f_i \cap e_{n-1} = f_i$ since $f_i \subset e_{n-1}$.

For $i = z+1$: $\langle v_{\pi(1)},\ldots,v_{\pi(z)},v_n\rangle = f_z \oplus v_n = f_{z+1}$, since $v_n \in f_{z+1} \setminus f_z$.

For $i > z+1$: $\langle v_{\pi(1)},\ldots,v_{\pi(z)},v_n,v_{\pi(z+1)},\ldots v_{\pi(i)}\rangle = f_{i-1} \oplus v_{\pi(i)}$, By induction, $v_{\pi(i)} \in (f_i \cap e_{n-1}) \setminus (f_{i-1} \cap e_{n-1})$. So, $v_{\pi(i)} \in f_i \setminus f_{i-1}$, as desired. ∎

As a corollary, given any two flags $e(\mu)$ and $f(\lambda)$, we can choose vectors $v_1,\ldots,v_n$ and a permutation π such that, for all i, $\langle v_1,\ldots,v_{\mu_1+\cdots+\mu_i}\rangle = e_i$ and $\langle v_{\pi(1)},\ldots,v_{\pi(\lambda_1+\cdots+\lambda_i)}\rangle = f_i$. Just expand e and f to flags of type 1^n: for example, expand $\mathbf{F}_q \subset \mathbf{F}_q^3$ to $\mathbf{F}_q \subset \mathbf{F}_q^2 \subset \mathbf{F}_q^3$. Then choose vectors and a permutation as above.

Lemma 2.2. *For any partitions μ, λ, the number of orbits of pairs $(e(\mu), f(\lambda))$ under GL_n is equal to the number of the orbits of pairs $(s(\mu), s(\lambda))$ in S_n. By $s(\mu)$, we mean a tabloid of type μ.*

Proof. By the previous lemma, we can choose vectors $v_1,\ldots v_n$ and a permutation π such that the $\langle v_1,\ldots,v_{\mu_1}\rangle = e(\mu)_1$, and so forth.

To each $(e(\mu), f(\lambda))$, associate the orbit of

$$\left(\begin{array}{ccc} \hline 1 & \cdots & \mu_1 \\ \mu_1+1 & \cdots & \mu_1+\mu_2 \\ & \vdots & \\ \mu_1+\cdots+\mu_{n-1}+1 & \cdots & n \end{array}, \begin{array}{ccc} \hline \pi(1) & \cdots & \pi(\lambda_1) \\ \pi(\lambda_1+1) & \cdots & \pi(\lambda_1+\lambda_2) \\ & \vdots & \\ \pi(\lambda_1+\cdots+\lambda_{n-1}+1) & \cdots & \pi(n) \end{array}\right).$$

This association is injective on orbits—if $(e(\mu), f(\lambda))$ and $(e'(\mu), f'(\lambda))$ both go to the same pair, then define g to be the change-of-basis matrix between the two different choices of vectors v: this will carry (e, f) to (e', f'), so they are in the same orbit.

It is well-defined on orbits—if g carries (e, f) to (e', f'), then if we choose $v_1, \dots v_n$ for e and f, then $gv_1, \dots, gv_n$ span e' and f' in the desired way, so (e', f') is associated with the same orbit of S_n.

It is surjective—we can achieve any orbit associated with π, λ, and μ by choosing $e = \langle \epsilon_1, \dots, \epsilon_{\mu_1} \rangle \subset \langle \epsilon_1, \dots, \epsilon_{\mu_1+\mu_2} \rangle \subset \cdots$, and $f = \langle \epsilon_{\pi(1)}, \dots, \epsilon_{\pi(\lambda_1)} \rangle \subset \langle \epsilon_{\pi(1)}, \dots, \epsilon_{\pi(\lambda_1+\lambda_2)} \rangle \subset \cdots$. □

Now that we have proved the second lemma, we can define what our irreducible character χ_μ of GL_n shall be.

Given a partition μ, put $\mu' = (\mu_1 + 1, \mu_2 + 2, \mu_3 + 3, \dots, \mu_n + n)$. Then define $\chi_\mu = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \rho(\mu' - \pi)$.

By $\mu' - \pi$ we mean the term-by-term subtraction $(\mu'_1 - \pi(1), \mu'_2 - \pi(2), \dots, \mu'_n - \pi(n))$.

Theorem 2.3. *The characters χ_μ are distinct irreducible characters of $GL_n(\mathbf{F}_q)$.*

Proof. First, note that the χ_μ are integral linear combinations of characters. So, we need to show $(\chi_\mu | \chi_\lambda) = \delta(\mu, \lambda)$, and that $\chi_\mu(1) > 0$.

Note that if r is a transitive, faithful representation, then $(r|1)$ is the number of its orbits: for each orbit $\{x_1, \dots, x_k\}$, $x_1 + \cdots + x_k$ is fixed by r, hence provides a stable one-dimensional subspace, and conversely so.

Using Lemma 2.2:

$$\begin{aligned}
(\rho(\mu)|\rho(\lambda)) &= (\rho(\mu) \otimes \rho(\lambda)|1) \\
&= \text{ \# orbits of flags } (e(\mu), e(\lambda)) \\
&= \text{ \# orbits of tabloids } (s(\mu), s(\lambda)) \\
&= (R_\mu \otimes R_\lambda | 1) \\
&= (R_\mu | R_\lambda).
\end{aligned}$$

We can disregard the complex conjugation since permutation representations are defined over $\mathbf{R}$.

We can apply this formula to calculate (χ_μ, χ_λ):

$$\begin{aligned}
(\chi_\mu|\chi_\lambda) &= \frac{1}{|GL_n|} \sum_{x \in GL_n} \sum_\pi \operatorname{sgn}(\pi) \rho(\mu' - \pi, x) \sum_\sigma \operatorname{sgn}(\sigma) \rho(\lambda' - \sigma, x) \\
&= \sum_{\pi,\sigma} \operatorname{sgn}(\pi) \operatorname{sgn}(\sigma) \left[\frac{1}{|GL_n|} \sum_{x \in GL_n} \rho(\mu' - \pi, x) \rho(\lambda' - \sigma, x) \right] \\
&= \sum_{\pi,\sigma} \operatorname{sgn}(\pi) \operatorname{sgn}(\sigma) (\rho(\mu' - \pi) | \rho(\lambda' - \sigma)) \\
&= \sum_{\pi,\sigma} \operatorname{sgn}(\pi) \operatorname{sgn}(\sigma) (R(\mu' - \pi) | R(\lambda' - \sigma)) \\
&= \left(\sum_\pi \operatorname{sgn}(\pi) R(\mu' - \pi) \middle| \sum_\sigma \operatorname{sgn}(\sigma) R(\lambda' - \sigma) \right) \\
&= (\phi(\mu)|\phi(\lambda)) \\
&= \delta(\mu, \lambda).
\end{aligned}$$

Here $\phi(\mu)$ is the character $\sum_\pi \mathrm{sgn}(\pi)R(\mu' - \pi)$ of S_n. The last line holds because the ϕ are irreducible in S_n. We will not prove this formula in this paper.

So, the χ's are irreducible and distinct. □

This computation is very powerful. It not only shows that the χ's are irreducible, but that they act just like the R's. They have the same formula in terms of permutation representations; χ's dimensions are given by similar formulas (see below). In fact, we might call χ a quantized version of R. In the limit as q goes to 1, the χ become R.

Let's look at an example.

Proposition 2.4. *This formula holds for the partition $\mu = (0, 1, 2)$ of $GL_n(F_q^3)$.*

Proof. We have

$$\begin{aligned}\phi_\mu &= \sum_{\pi \in S_3} \rho(1, 3, 5 - \pi)\\ &= \rho(0, 1, 2) - \rho(0, 0, 3)\end{aligned}$$

The permutation representation $\rho(0, 1, 2)$ is just the permutation representation on the standard flag spanned by $\epsilon_1, \epsilon_2, \epsilon_3$, and $\rho(0, 0, 3)$ is trivial (there is only one flag of type $(0, 0, 3)$). If we check the character with the inner product, we see that this representation is, indeed, simple. □

We have postponed the proof that $\chi(\mu, 1) > 0$, as it is technical. This follows in a separate theorem.

Theorem 2.5. *The χ_μ are* bonafide *characters, not just virtual characters.*

Proof. Now, we need to show $\chi(\mu, 1) > 0$. This depends on two technical calculations:

First, let's compute the dimension of $\rho(\mu)$: this is the number of flags of type μ.

Note for every orthonormal choice of basis vectors $\epsilon_1, \ldots \epsilon_n$, we can associate the corresponding flag $\langle \epsilon_1, \ldots, \epsilon_{\mu_1} \rangle \subset \langle \epsilon_1, \ldots, \epsilon_{\mu_1+\mu_2} \rangle \subset \cdots \subset \langle \epsilon_1, \ldots, \epsilon_n \rangle$.

If we denote by $[i]!$ the number of orthonormal ways of choosing a basis for an i-dimensional vector space, we have $[n]!$ flags so far.

However, this correspondence isn't unique. Any such choice of basis vectors $e_1, \ldots, e_{\mu_1}$ gives us the same V_1. Similarly, given our choice of $e_1, \ldots, e_{\mu_1}$, we can choose orthonormal vectors for V_2/V_1, i.e., $[\mu_2]!$ more choices.

Continuing this way, we see the dimension of $\rho(\mu)$ is

$$\frac{[n]!}{[\mu_1]! \cdots [\mu_n]!}.$$

Now, let's calculate $[i]!$. We can choose an orthonormal basis for V of dimension i by first choosing any unit vector v in V, then choosing a vector in $v^\perp$, and so on. So $[i]! = [i][i-1] \ldots [1]$, where $[i]$ is the number of unit vectors in an i dimensional vector space of $\mathbf{F}_q$.

In $\mathbf{F}_q^i$, there are $q^i - 1$ non-zero vectors, each of which has $q - 1$ distinct multiples, for $(q^i - 1)/(q - 1)$ unit vectors. So $[i] = q^{i-1} + q^{i-2} + \cdots + 1$.

(A bit of notation: We define $[i]! = 0$ for $i < 0$.)

Note the correspondence of this formula with the formula for the dimension of $R(\mu)$, ρ's S_n-counterpart.

Now, we can calculate $\chi_\mu(1)$:

$$\begin{aligned}
\chi_\mu(1) &= \sum_\pi \operatorname{sgn}(\pi)\rho(\mu' - \pi, 1) \\
&= \sum_\pi \operatorname{sgn}(\pi)\frac{[n]!}{([\mu'_1 - \pi_1]!)\cdots([\mu'_n - \pi_n]!)} \\
&= [n]! \det\left|([\mu'_i - j]!)^{-1}\right| \\
&= [n]! \det\left|\frac{1}{[\mu'_i - j][\mu'_i - (j+1)]\cdots[1]}\right| \\
&= [n]! \det\left|\frac{[\mu'_i]\cdots[\mu'_i - (j-1)]}{[\mu'_i]!}\right| \\
&= \frac{[n]!}{[\mu'_1]!\cdots[\mu'_n]!}\det\left|[\mu'_i][\mu'_i - 1]\cdots[\mu'_i - (j-1)]\right| \\
&= \frac{[n]!}{[\mu'_1]!\cdots[\mu'_n]!}\det\left|[\mu'_i]\frac{[\mu'_i]-[1]}{q}\cdots\frac{[\mu'_i]-[j-1]}{q^{j-1}}\right|
\end{aligned}$$

The last line follows from the identity $[a-b] = ([a]-[b])/q^b$.

To calculate this determinant, we'll substitute indeterminates X_i for $[\mu'_i]$:

$$\det\left|X_i \ \frac{X_i - [1]}{q}\cdots\frac{X_i - [j-1]}{q^{j-1}}\right|.$$

The degree of this is $n(n-1)/2$ as a polynomial in the X_i: row j's cells each have degree $j-1$, so when we calculate the determinant, each summand will have degrees $0+1+2+\cdots+(n-1)$. Also, if $X_i = X_k$, then columns i and k will be the same, and the determinant will vanish. So, this determinant is a constant multiple of the van der Monde determinant $\prod_{i<j}(X_j - X_i)$.

Let's find what this constant multiple is. We need to find the coefficient of one term. The coefficient of the $X_2X_3^2X_4^3\cdots X_n^{n-1}$ term is this: in taking the determinant, we must choose the cells $(1,0)$, $(2,1)$, and so on, since column determines which X_i and row determines its degree. This gives coefficients

$$1 \times 1/q \times 1/q^{1+2} \times 1/q^{1+2+3} \times \cdots$$

which equals $q^{-n(n-1)(n-2)/3}$, whereas in the van der Monde determinant, this term would have coefficient unity. So

$$\det\left|X_i \ \frac{X_i - [1]}{q}\cdots\frac{X_i - [j-1]}{q^{j-2}}\right| = q^N\prod_{i<j}(X_j - X_i).$$

Whew! Almost done.

Let's plug this into our formula for $\chi(\mu, 1)$:

$$\chi(\mu,1) = \frac{[n]!q^N\prod_{i<j}([\mu'_j]-[\mu'_i])}{[\lambda_1]!\cdots[\lambda_n]!}$$

This is positive. The $[x]!$ terms are all positive, the q^N term is positive, and, since the μ'_i are increasing (since the μ_i are), each $([\mu'_j]-[\mu'_i])$ term is positive too. □

It's beyond the scope of this paper to prove that

$$\phi(\mu) = \sum_{\pi} \operatorname{sgn}(\pi) R(\mu + (1, 2, \ldots, n) - \pi)$$

is an irreducible character of S_n.

We can point the way to this result, by finding a formula for S_n's irreducibles.

3. The irreducible representations of S_n

The natural place to start is with the permutation representation M_μ for a partition μ. (The notation M_μ means we consider it as a $\mathbf{C}[S_n]$-module, with the natural action of S_n on tabloids.) As a module, M_μ has basis $\{t\}$, where t are tabloids of type μ. We have seen that, convenient though it be, M_μ is not simple. For example, if μ is the trivial partition $(1, 1)$ in S_2, then $\{1\}\{2\} + \{2\}\{1\}$ gives a one-dimensional fixed submodule. However, we can find a good submodule of M_μ which is irreducible.

Let us define e_t, for any tableau t of type μ, to be

$$e_t = \{t\} \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\pi.$$

Here, C_t is the column-stabilizer of t, the set of permutations which don't move letters from one column of t to another. So, for the tableau $\begin{smallmatrix}1 & \\ 2 & 3\end{smallmatrix}$, the column stabilizer is $\{(1\ 2), 1)\}$. The addition of tabloids here is purely formal.

e_t is called a polytabloid. It is crucial in this definition that, even though e_t is a tabloid, t itself is a *tableau*—that is, the order of the letters in t's rows is important, but not in e_t's.

Here is an example. If $t = \begin{smallmatrix}1 & \\ 2 & 3\end{smallmatrix}$, then $e_t = \overline{\begin{smallmatrix}1 & \\ 2 & 3\end{smallmatrix}} - \overline{\begin{smallmatrix}2 & \\ 1 & 3\end{smallmatrix}}$. That is, given a tableau t, we permute the columns in all possible ways, and take the sum of these tabloids taking the sign of our permutations into account, and then draw a line on top at the end.

Now, define M'_μ to be the submodule spanned by the polytabloids. This is called the Specht submodule.

First, we prove some lemmata.

Lemma 3.1. *M'_μ is generated by any polytabloid.*

Proof. This follows from the fact that $e_t\pi = e_{t\pi}$, which in turn follows from the fact that $C_{t\pi} = \pi^{-1}C_t\pi$. □

Lemma 3.2. *Let t and t' be tableaux of type μ such that $\{t'\}\sum_{\pi \in C_t}(\operatorname{sgn}\pi)\pi \neq 0$. Then $\{t'\}\sum_{\pi \in C_t}(\operatorname{sgn}\pi)\pi = \pm e_t$.*

Proof. Let a and b be any two numbers in the same row of t'. Say a and b were in the same column of t. Then $\{1, (a\ b)\}$ would be a subgroup of C_t. Choosing $g_1, \ldots, g_l$ to be its coset representatives, we would have

$$\begin{aligned}
\{t'\} \sum_{\pi \in C_t} (\operatorname{sgn}\pi)\pi &= \{t'\}\left((e - (a\ b))(g_1 + \cdots + g_k)\right) \\
&= (\{t'\} - \{t'\}(a\ b))(g_1 + \cdots + g_k) \\
&= (\{t'\} - \{t'\})(g_1 + \cdots + g_k) \\
&\text{(Since } a, b \text{ are in the same row, } \{t'\}(a, b) = \{t'\}\text{—it's a } \textit{tabloid.}) \\
&= 0.
\end{aligned}$$

So, for every i, the numbers in the ith row of t' lie in in different columns of t. Since $\{t'\}$ is a tabloid, this means we can rearrange $\{t'\}$ such that for every letter k, k lies in the same column of $\{t'\}$ as in t. In other words, $\{t'\} = \{t\}\,\rho$ for some $\rho \in C_t$. So

$$\begin{aligned}
\{t'\} \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\pi &= \{t\}\,\rho \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\pi \\
&= \{t\} \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\pi\rho \\
&= \pm \{t\} \sum_{\pi \in C_t} (\operatorname{sgn} \pi\rho)\pi\rho \\
&= \pm \{t\} \sum_{\pi\rho \in C_t} (\operatorname{sgn} \pi\rho)\pi\rho, \text{ since } \rho \in C_t \\
&= \pm e_t.
\end{aligned}$$

□

So, for all tableaux t', $t' \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\pi$ is a multiple $(0, 1, \text{ or } -1)$ of e_t.

Lemma 3.3. *If $u \in M_\mu$ and t is a tableau of type μ, then $u \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\pi$ is a multiple of e_t.*

Proof. By definition of M_μ, u is a linear combination of tabloids $\{t'\}$. By the previous lemma, $t' \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\pi$ are each multiples of e_t. □

Now, define a bilinear form $\langle \mid \rangle$ on M_μ by:

$$\langle \{t_1\} \mid \{t_2\} \rangle = \delta(\{t_1\}, \{t_2\}).$$

This form is visibly symmetric and S_n-invariant.

Lemma 3.4. *If $U \leq M_\mu$, then either $M'_\mu \subset U$ or $U \subset {M'_\mu}^{\perp}$.*

Proof. Let $u \in U$. Then, by the previous lemma, for all tableaux t, $u \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\pi = k_{u,t} e_t$, for $k_{u,t} \in \mathbf{C}$. There are two cases:

- For some tableau t and $u \in U$, $k_{u,t} \neq 0$. Then we have

$$e_t = \frac{u \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\pi}{k_{u,t}},$$

  so $e_t \in U$ (U is a submodule). Since M'_μ is generated by e_t, it must be that $M'_\mu \subset U$.

- For all tableaux t and $u \in U$, $k_{u,t} = 0$. Then for all u, t

$$\begin{aligned}
0 &= u \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\pi \\
&= \langle u \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\pi | \{t\}\rangle \\
&= \sum_{\pi \in C_t} \langle u\pi | (\operatorname{sgn} \pi) \{t\}\rangle \\
&= \sum_{\pi \in C_t} \langle u | (\operatorname{sgn} \pi)\pi^{-1} \{t\}\rangle \quad (\text{since } \langle \ | \ \rangle \text{ is } S_n\text{-invariant.}) \\
&= \sum_{\pi \in C_t} \langle u | (\operatorname{sgn} \pi^{-1})\pi^{-1} \{t\}\rangle \\
&= \left\langle u \middle| \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\pi \{t\} \right\rangle \\
&= \langle u | e_t \rangle.
\end{aligned}$$

  Since the e_t generate M'_μ, it must be that $U \subset {M'_\mu}^\perp$.

□

Theorem 3.5. *M'_μ is irreducible.*

Proof. By the previous theorem, the only submodule of M_μ contained in M'_μ is M'_μ; M'_μ has no nontrivial submodules, so is irreducible. □

Note that M_μ is also derived from inducing the trivial representation on $S_{\mu_1} \times \cdots \times S_{\mu_n}$. So this theorem tells us how to get S_n's irreducible characters from characters on lower symmetric groups, a kind of induction.

4. An example

While the theorem of Section 3 tells us a lot about the irreducibles of S_n, it doesn't give the powerful theorem we used in Section 2. Let's return to this formula.

$$\sum_\pi \operatorname{sgn}(\pi) R(\mu' - \pi) \text{ is irreducible.}$$

Since we have actually classified the irreducibles of S_n, we can rewrite this:

$$M'_\mu = \sum_\pi \operatorname{sgn}(\pi) M_{\mu' - \pi}.$$

Or, equivalently:

$$M'_\mu \oplus \bigoplus_{\pi \in S_n \setminus A_n} M_{\mu' - \pi} \cong \bigoplus_{\pi \in A_n} M_{\mu' - \pi}.$$

I won't prove this formula—the proof involves a lot of tricky combinatorics. However, I will show how this works in the elementary case of $\mu = (1,1)$. This demonstrates the key algebraic concepts.

Theorem 4.1.

$$M'_{\{(1,1)\}} = \sum_{\pi \in S_2} \operatorname{sgn}(\pi) M_{\{(1,1)+(1,2)-\pi)\}}.$$

Proof. We wish to show

$$M'_{\{(1,1)\}} \oplus M_{\{(0,2)\}} \cong M_{\{(1,1)\}}.$$

$M'_{\{(1,1)\}}$ is generated by the polytabloids $\{1\}\{2\} - \{2\}\{1\}$ and $\{2\}\{1\} - \{1\}\{2\}$. $M_{\{(0,2)\}}$ is generated by the tabloid $\{1\ 2\}$. $M_{\{(1,1)\}}$ is generated by the tabloids $\{1\}\{2\}$ and $\{2\}\{1\}$. For the isomorphism, send

$\{1\}\{2\}$ to $\{1\}\{2\} - \{2\}\{1\} + \{1\ 2\}$, and

$\{2\}\{1\}$ to $\{2\}\{1\} - \{1\}\{2\} + \{1\ 2\}$.

This respects all the relations on S_n.

The point of this example is when we pass from $M_{(1,1)}$ to $S_{(1,1)}$, we lose the distinction between $\{1\}\{2\}$ and $\{2\}\{1\}$—in $M'_{(1,1)}$, they are linear multiples. By adding the $\{1\ 2\}$ as a direct summand, we are able to fix sign for these again. At the same time, since $\{1\ 2\}$, as a tabloid, is fixed by all the permutations fixing $\{1\}\{2\}$ and $\{2\}\{1\}$, adding it doesn't interfere with any of the algebraic relations between S_n and $S_{(1,1)}$. □

References

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