# APPLICATIONS OF REPRESENTATION THEORY TO FINITE GROUP STUDY

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Abstract. Particular branches of representation theory, most notably the concept of exceptional characters and their importance in the study of trivial intersection subsets, leads to a succinct classification of finite groups of odd order satisfying the relation

$$1 \neq x \in G \implies C_G(x) \text{ is abelian.}$$

Here we use $C_G(x)$ to denote the centralizer of $x$ in $G$. We shall explore this classification in this exposition.

## 1. Frobenius Groups

**Definition.** Let $H$ be a proper, nontrivial subgroup of a group $G$. Then $H$ is called a **Frobenius subgroup** of $G$ iff:

$$x \notin H \implies (H \cap H^x) = 1,$$

where $H^x$ is the conjugate subgroup $x^{-1}Hx$. A group $G$ which contains a Frobenius subgroup is called a **Frobenius group**.

**Lemma 1.1.** *Let $H$ be a Frobenius subgroup of a Frobenius group $G$. $\forall$ nonidentity $x \in H$, $C_G(x) \subset H$.*

*Proof.* $y \in C_G(x) \implies x^y = x$, and so $1 \neq x \in (H \cap H^y) \implies y \in H$ by the definition of a Frobenius subgroup. □

We bring up the concept of Frobenius groups not only for the integral role they play in analyzing groups of odd order which satisfy the above centralizer condition, but also for their value in demonstrating the importance of representation theory to the study of finite group structure, as the following theorem demonstrates.

**Theorem 1.2.** *Let $H$ be a Frobenius subgroup of a Frobenius group $G$. We define $N$ to be the set of elements in $G$ not conjugate to any nonidentity element of $H$. Then $|N| = |G : H|$. Furthermore, $N \lhd G$ and $G = NH$.*

*Proof.* Let us denote by $C_1, \dots, C_k$ the conjugacy classes of $H$, so that $C_1 = \{1\}$. Define for $i > 1$

$$C_i^* = \{x \in G \mid x^y \in C_i \text{ for some } y \in G\}.$$

(Here we use $x^y$ to denote the conjugate element $y^{-1}xy$.) By Lemma 1.1, $C_G(x) \subset H$ for nonidentity $x \in H$, and so $C_H(x) = C_G(x)$. $x \in C_i$ for some $i > 1$. But

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since $|G|/|C_G(x)| = |C_i^*|$ and $|H|/|C_H(x)| = |C_i|$, if $|G : H| = n$ then $|C_i^*| = n|C_i|$. Since

$$N = G - \bigcup_{i>1} C_i^*,$$

$|N| = n|H| - n\left(|H| - 1\right) \implies |N| = |G : H|$.

To see that $N$ is indeed a normal subgroup of $G$, set $C_1^* = N$, and let us first show that the class function

$$\phi^*(C_i^*) = \phi(C_i)$$

is an irreducible character on $G$, where $\phi$ an arbitrary irreducible character of $H$ over $\mathbf{C}$. Recall that for $x \in G$, the induced character $\phi^G$ is defined as

$$\phi^G(x) = \frac{1}{|H|}\sum_{y\in G}\phi^o(y^{-1}xy),$$

where

$$\phi^o(z) = \phi(z) \text{ for } z \in H, \text{ and } \phi^o(z) = 0 \text{ for } z \notin H.$$

We see, therefore, that

$$\phi^* - \phi(1)1_G = \left(\phi - \phi(1)1_H\right)^G,$$

where $1_G$ and $1_H$ represent the trivial characters on $G$ and $H$, respectively. This follows since both sides of the above equation produce 0 when applied to $x \in C_1^*$, while for $x \in C_i^*$, $i > 1$, $\phi^*(x) = \phi^G(x)$ and $1_G(x) = (1_H)^G(x)$ since $|C_i^*| = n|C_i|$. Thus, $\phi^*$ is a generalized character of $G$, by which we mean an integral linear combination $\sum a_i\chi_i$ for $\chi_i$ irreducible characters of $G$. Since $\left(\phi^*, \phi^*\right)_G = \sum a_i^2$, $\phi^*$ is itself irreducible on $G$ iff $\left(\phi^*, \phi^*\right)_G = 1$ and $\phi^*(1) > 0$. But $\phi^*(1) = \phi(1) > 0$ since $\phi$ is irreducible on H, and $\left(\phi^*, \phi^*\right)_G = (1/|G|)\sum_i |\phi^*(C_i^*)|^2|C_i^*| = (1/|H|)\sum_i |\phi(C_i)|^2|C_i| = \left(\phi, \phi\right)_H = 1$, again by the irreducibility of $\phi$.

Now, since $\phi^*(n) = \phi^*(1)$ for $n \in N$ by definition, $N$ is contained in the kernel of the representation with character $\phi^*$ for any given irreducible character $\phi$ of $H$ over $\mathbf{C}$. But since $\mathbf{C}$ is a field of characteristic 0, given any nonidentity $x \in H$ $\exists$ an irreducible representation $\rho$ of $H$ s.t. $x \notin \ker\rho$. (This is a highly nontrivial result, stated in [2, Proposition 1.4 of Chapter 6], which requires Clifford's Theorem, among other things, for its proof.) Thus, for any $x \in C_i^*$, $i > 1$, $\exists$ an irreducible character $\phi$ of $H$ s.t. $x$ is not contained in the kernel of the representation with character $\phi^*$. Hence, $N$ is precisely the intersection of the kernels of all representations with character $\phi^*$ corresponding to some irreducible character $\phi$ of $H$. Thus, $N \lhd G$ as claimed.

Since $N \lhd G$, $N \cap H = \{1\}$, and $|N| = |G : H|$, it follows by elementary group theory that $G = NH$. (See [1, Section 4 of Chapter 1], if necessary, for details.) □

**Definition.** The normal subgroup $N$ of the Frobenius group $G$ in Theorem 1.2 is called the **Frobenius kernel** of $G$.

It turns out that the Frobenius kernel $N$ of a Frobenius group $G$ necessarily satisfies a particular centralizer condition, and that in fact this condition is sufficient to classify any proper, nontrivial normal subgroup $N$ of an arbitrary group $G$ as a Frobenius kernel, and $G$ as a Frobenius group. This fact is subsumed in the following theorem, which will prove vital in later applications.

**Theorem 1.3.** *Let $N$ be a Frobenius kernel of the Frobenius group $G$. $N$ is a nontrivial, proper, normal subgroup of $G$ satisfying*

$$1 \neq x \in N \implies C_G(x) \subset N. \tag{1}$$

*Conversely, a nontrivial, proper, normal subgroup $N$ of an arbitrary group $G$ which satisfies* (1) *is a Frobenius kernel, and $G$ is a Frobenius group.*

*Proof.* We begin by proving the necessity of (1). Let $G$ be a Frobenius group with Frobenius kernel $N$. Since a Frobenius subgroup $H$ of $G$ is proper and nontrivial by definition, it follows that since $G = NH$, $N$ must be proper and nontrivial as well. For $i > 1$, take $x \in C_i^*$, where the $C_i^*$ are defined as before with respect to the conjugacy classes $C_i$ of a Frobenius subgroup $H$ of $G$. $x^y \in C_i$ for some $y \in G$, and hence $x \in H^{y^{-1}}$. But it can be trivially verified that $H^{y^{-1}}$ satisfies the properties specified in (1), and so $H^{y^{-1}}$ is itself a Frobenius subgroup of $G$. By Lemma 1.1, $C_G(x) \subset H^{y^{-1}}$. But by definition, $(N - \{1\})$ contains no elements conjugate to any element of $H$, and so $(N - \{1\}) \cap H^{y^{-1}} = \emptyset$. Thus, $\forall x \in C_i^*$, $i > 1$, $(N - \{1\}) \cap C_G(x) = \emptyset$, i.e. no nonidentity element of $N$ commutes with any element of $C_i^*$ for $i > 1$. We conclude that given nonidentity $x \in N$, $C_G(x) \subset N$.

Now we show that (1) is sufficient to characterize a proper, nontrivial subgroup $N \lhd G$ as a Frobenius kernel of the Frobenius group $G$. Various group theoretic arguments show that since $C_N(x) = 1$ for $x \notin N$, $|N|$ is relatively prime to $|G : N|$, and hence $\exists$ a subgroup $H \subset G$ s.t. $NH = G$ and $N \cap H = 1$. (The means to prove this assertion, though based on fundamental tenets of group theory, lies far astray of the scope of this exposition. See [2, page 280] for details.) In fact, $H$ is a Frobenius subgroup of $G$. To see this, we will show that, for $x \in G$,

$$H \cap H^x \neq 1 \implies x \in H,$$

from which the assertion follows. $H$ must be nontrivial and proper, since $N$ is nontrivial and proper and $G = HN$. If $H \cap H^x \neq 1$, $\exists$ some nonidentity $h \in H$ s.t. $h^x \in H$ as well. Since $G = HN$, $x = h'n$ for $h' \in H$, $n \in N$. Now $\left(h^{h'}\right)^{-1} n^{-1} h^{h'} n \in N$ since $N$ is a normal subgroup of $G$. However, $\left(h^{h'}\right)^{-1} n^{-1} h^{h'} n = \left(h^{h'}\right)^{-1} h^{h'n} = \left(h^{h'}\right)^{-1} h^x \in H$ as well, and since $N \cap H = 1$, we see that $\left(h^{h'}\right)^{-1} n^{-1} h^{h'} n = 1$, i.e. $h^{h'}$ and $n$ commute. But since $h \neq 1$, $h^{h'} \notin N$, and by (1), therefore, $n = 1$. So $x = h' \in H$, as we desired. Since $H$ is a Frobenius subgroup and hence $G$ a Frobenius group, we can then easily verify that $N$ satisfies the conditions outlined in Theorem 1.2 and conclude that $N$ is the Frobenius kernel of $G$. □

## 2. Trivial Intersection Subsets and Exceptional Characters

**Definition.** Let $K$ be a subset of a subgroup $H$ of a group $G$. $K$ is called a **trivial intersection subset** with respect to $G$ and $H$ (or, in shorthand, a ***TI*-subset** in $G$) iff:

(i) $N_G(K) = H$, where $N_G(K)$ represents the normalizer of $K$ in $G$;
(ii) Elements of $K$ which are conjugate in $G$ are likewise conjugate in $H$;
(iii) $\forall$ nonidentity $x \in K, C_G(x) \subset H$.

Trivial intersection subsets are in fact characterized succinctly by a condition similar to that which defines Frobenius subgroups. This characterization will oftentimes prove more useful that the rather cumbersome definition above, and so we state it below for convenience.

**Lemma 2.1.** *Let $K$ be a nonempty, nontrivial subset of a group $G$, and set $H = N_G(K)$. $K$ is a* TI-*subset in $G$ iff:*

$$(K \cap K^x) \subset \{1\} \quad \forall x \in (G - H). \tag{2}$$

*Proof.* Assume $K$ is a *TI*-subset in $G$, and suppose $\exists\ x \in G$ s.t. nonidentity $k \in (K \cap K^x)$. $k \in K$ and $k = (k')^x$ for some $k' \in K$. So $k$ and $k'$ are conjugate in $G$; by property (ii) of the above definition, they are conjugate in $H$, i.e. $\exists\ h \in H$ s.t. $k = (k')^h$. So $(k')^x = (k')^h$, implying that $xh^{-1} \in C_G(k')$. But since $k$ is nonidentity, so too is its conjugate $k'$, and so by property (iii), $C_G(k') \subset H$. Thus, $xh^{-1} \in H \implies x = (xh^{-1})h \in H$, proving that (2) must hold if $K$ is a *TI*-subset in $G$.

Conversely, assume $K$ satisfies (2). For nonidentity $x \in K$, $y \in C_G(x) \implies x^y = x$, and so nonidentity $x \in (K \cap K^y) \implies y \in H$ by (2). Thus, property (iii) of the definition holds. If $k' = k^x$ for nonidentity $k, k' \in K$ and $x \in G$, (2) implies that $x \in H$, and so property (ii) of the definition holds. Property (i) is satisfied by definition. □

For any subset $K$ contained in a subgroup $H$, let us denote by $gc(H)$ the set of all generalized characters of $H$, and define

$$gc(H;K) = \{\theta \in gc(H) \mid \theta(x) = 0 \quad \forall x \in (H - K)\}.$$

The following lemma displays a couple of important properties possessed by the elements of $gc(H;K)$ when $K$ is a *TI*-subset.

**Lemma 2.2.** *Let $K$ be a* TI-*subset with respect to $G$ and $H$, and let $\theta \in gc(H;K)$. Then*

$$\theta^G(x) = \theta(x) \quad \forall x \in (K - 1).$$

*Furthermore, if $\theta(1) = 0$, then for $\phi \in gc(H;K)$ we must have*

$$\left(\theta^G, \phi^G\right)_G = (\theta, \phi)_H.$$

*Proof.* Let nonidentity $x \in K$. We saw in our proof of Lemma 2.1 that if $x^y \in K$ for some $y \in G$, $y \in H$ by necessity. So, coupling the fact that $\theta \in gc(H;K)$ vanishes on $(H - K)$ with the formula for computing the induced character, we see that $\theta^G(x) = (1/|H|)\sum_{y\in G}\theta^o(y^{-1}xy) = (1/|H|)\sum_{y\in H}\theta(x) = \theta(x)$, proving the first equation of Lemma 2.2.

Now let $\theta(1) = 0$. We write $\left(\theta^G\right)_H = \theta + \theta_0$, where $\theta_0$ is a generalized character on $H$. Since $\theta(1) = 0$, $\theta^G(1) = 0$ by definition, and so $\theta = \theta^G$ on all of $K$, i.e. $\theta_0$ vanishes on $K$. Since $\phi \in gc(H;K)$ vanishes on $(H - K)$, $(\theta_0, \phi)_H = 0$. Coupling this fact with Frobenius reciprocity yields $\left(\theta^G, \phi^G\right)_G = \left(\left(\theta^G\right)_H, \phi\right)_H = (\theta + \theta_0, \phi)_H = (\theta, \phi)_H$, and so the second equation of Lemma 2.2 is proved. □

Using the lemmas we have developed about a *TI*-subset $K$ contained in a subgroup $H$ of a group $G$, we now establish a correspondence between certain irreducible characters of $H$ and a set of irreducible characters of $G$.

**Theorem 2.3.** *Let $K$ be a* TI-*subset with respect to $G$ and $H$. Suppose $\exists$ a set of irreducible characters of $H$, $\{\phi_1, \ldots, \phi_w\}$, satisfying $w \geq 2$ and*

$$\phi_i(x) = \phi_j(x) \quad \forall i, j \text{ and } \forall x \in \{H - K, 1\}.$$

*Then $\exists$ a set of irreducible characters of $G$, $\{\chi_1, \ldots, \chi_w\}$, and a sign $\epsilon = \pm 1$ satisfying*

$$\phi_i^G = \epsilon \chi_i + \Delta,$$

*where $\Delta$ is independent of $i$ and either $\Delta \equiv 0$ on $G$ or $\Delta$ is a character of $G$.*

*Proof.* Suppose $\{\phi_1, \ldots, \phi_w\}$ exists as above. Fix any $i$ and $j$, and set $\theta = (\phi_i - \phi_j)$. Note that $\theta \in gc(H; K)$ s.t. $\theta(1) = 0$; thus, by Lemma 2.2, $\left(\theta^G, \theta^G\right)_G = (\theta, \theta)_H = (\phi_i - \phi_j, \phi_i - \phi_j)_H = 2$ due to the irreducibility of $\phi_i$ and $\phi_j$. Also, $\theta(1) = 0 \implies \theta^G(1) = 0$, so we see that $\theta^G$ is the difference of two irreducible characters on $G$.

If $w = 2$, then it follows automatically that $\theta^G = (\phi_1 - \phi_2)^G = \chi_1 - \chi_2$, where $\chi_1$ and $\chi_2$ are suitably chosen irreducible characters on $G$.

For $w = 3$, note that $\left((\phi_1 - \phi_2)^G, (\phi_1 - \phi_3)^G\right)_G = 1$. So denote the irreducible character of $G$ contained by both $(\phi_1 - \phi_2)^G$ and $(\phi_1 - \phi_3)^G$ by $\chi_1$, and denote its multiplicity in each by $\epsilon = \pm 1$. We see therefore that, following the proof for the case $w = 2$, we are left with $(\phi_1 - \phi_2)^G = \epsilon(\chi_1 - \chi_2)$ and $(\phi_1 - \phi_3)^G = \epsilon(\chi_1 - \chi_3)$ for suitably chosen irreducible characters $\chi_2$ and $\chi_3$ on $G$. It also follows that $(\phi_2 - \phi_3)^G = (\phi_1 - \phi_3)^G - (\phi_1 - \phi_2)^G = \epsilon(\chi_2 - \chi_3)$.

For $w \geq 4$, the preceding paragraph allows us to again derive $(\phi_1 - \phi_2)^G = \epsilon(\chi_1 - \chi_2)$, $(\phi_1 - \phi_3)^G = \epsilon(\chi_1 - \chi_3)$, and $(\phi_2 - \phi_3)^G = \epsilon(\chi_2 - \chi_3)$ for suitably chosen irreducible characters $\chi_1$, $\chi_2$, and $\chi_3$ on $G$. So take $(\phi_1 - \phi_i)^G$ for $i \geq 4$. Since $\left((\phi_1 - \phi_i)^G, (\phi_1 - \phi_j)^G\right)_G = 1$ for both $j = 2$ and $j = 3$, we see that $(\phi_1 - \phi_i)^G$ must contain $\chi_1$ with multiplicity $\epsilon$. For if not, $(\phi_1 - \phi_i)^G = -\epsilon(\chi_2 + \chi_3) \implies \chi_2(1) + \chi_3(1) = 0$, a contradiction since $\chi_2$ and $\chi_3$ are both irreducible on $G$. So we may write $(\phi_1 - \phi_i)^G = \epsilon(\chi_1 - \chi_i)$ for $i \geq 4$, where the $\chi_i$ are suitably chosen irreducible characters over $G$, and by induction we can obtain $(\phi_i - \phi_j)^G = \epsilon(\chi_i - \chi_j)$ $\forall i, j$.

Since we have shown that $(\phi_i - \phi_j)^G = \epsilon(\chi_i - \chi_j)$ $\forall i, j$ for some set $\{\chi_1, \ldots, \chi_w\}$ of irreducible characters on $G$, we are left only to re-express this relation as $\phi_i^G - \epsilon\chi_i = \phi_j^G - \epsilon\chi_j = \Delta$, where we note that $\Delta$ is a character on $G$ (or $\Delta \equiv 0$), and that $\Delta$ is independent of $i$. Thus, $\phi_i^G = \epsilon\chi_i + \Delta$, as desired. □

**Definition.** The set $\{\chi_1, \ldots, \chi_w\}$ of irreducible characters on $G$ derived in Theorem 2.3 are termed the **exceptional characters** associated with the set $\{\phi_1, \ldots, \phi_w\}$ of irreducible characters on $H$.

Exceptional characters prove vital in proving our main theorem due to the following lemma, whose proof, though perhaps within our grasp, contains too much gory algebraic detail to be very insightful to prove. We refer the interested reader to Suzuki's development of this proposition.

**Lemma 2.4.** *Let $K_1, \ldots, K_s$ be the representatives of the conjugacy classes of maximal abelian subgroups in a group $G$. If every nontrivial irreducible character of $G$ is an exceptional character for some $K_i$, then $N_G(K_i) = K_i$ for some $i$.*

*Proof.* This is equivalent to what follows part (d) of Example 1 in [2, Section 2 of Chapter 6], and relies heavily upon size arguments developed in Theorems 2.12 and 2.13 of the same. □

We state one final lemma that will aid us greatly in the proof of our main theorem.

**Lemma 2.5.** *Let $K$ be a nontrivial abelian subgroup of a group $G$ which satisfies*

$$1 \neq x \in K \implies C_G(x) = K. \tag{3}$$

*Notationally, we set $H = N_G(K)$, $l = |H : K|$, and $w = (|K| - 1)/l$. Then $K$ is a* TI*-subset with respect to $G$ and $H$. Furthermore, if $K \neq H$, then $H$ is a Frobenius group with Frobenius kernel $K$. The nontrivial linear characters on $K$ induce precisely $w$ irreducible characters $\phi_1, \ldots, \phi_w$ on $H$ of like degree, all of which vanish outside of $K$.*

*Proof.* To verify that $K$ is a *TI*-subset with respect to $G$ and $H$, we show that $1 \neq x \in (K \cap K^y) \implies y \in H$, from which (2) of Lemma 2.1 follows. So if $1 \neq x \in (K \cap K^y)$ for some $y \in G$, then $1 \neq x^{y^{-1}} \in K$ as well, and so $C_G(x) = C_G(x^{y^{-1}}) = K$. But then $K^y = \left(C_G(x^{y^{-1}})\right)^y = C_G(\left(x^{y^{-1}}\right)^y) = C_G(x) = K$, and so by definition $y \in N_G(K) = H$.

Now assume $K \neq H$. $K$ is a nontrivial, proper, normal subgroup of $H$; furthermore, $K$ satisfies the conditions of (1) in Theorem 1.3, so it follows immediately that $K$ is a Frobenius kernel of the Frobenius group $H$. Let $L$ be a Frobenius subgroup of $H$ s.t. $H = KL$ and $K \cap L = \{1\}$. We see that $|L| = |H : K| = l$. Some group theoretic arguments can be employed to show that, since an element $x \in L$ has order prime to $|K|$, the conjugate character $\rho^x$ of a nontrivial linear character $\rho$ on $K$ is distinct $\forall\ x \in L$, and that in fact this set of $l$ characters comprises all conjugate characters of $\rho$. Furthermore, Clifford's Theorem tells us that, given an irreducible character $\phi$ on $H$ s.t. $\phi_K$ contains $\rho$, $\phi_K$ in fact contains $\rho^x\ \forall\ x \in L$, and hence $\rho^H = \phi$. (These arguments are fleshed out in [2, p. 287]; Clifford's Theorem can be found in [2, p.264].) More importantly from our perspective, since there are $|K| - 1$ nontrivial linear characters on the abelian group $K$, there must be $(|K| - 1)/l = w$ irreducible characters on $H$ induced by the nontrivial linear characters on $K$. Label these $\phi_1, \ldots, \phi_w$. Since $K \lhd H$, no element of $(H - K)$ is conjugate to any element of $K$, and so $\phi_i$ must vanish outside of $K$ by the definition of the induced character $\forall i$. Also, $\phi_i(1) = |H : K| = l$, so the induced characters are of like degree. □

## 3. Finite Group Study Applications

**Theorem 3.1.** *Let $G$ be a group of odd order satisfying*

$$1 \neq x \in G \implies C_G(x) \text{ is abelian.} \tag{4}$$

*Then $G$ is either an abelian group or a Frobenius group.*

*Proof.* Let us take a set of representatives $K_1, \ldots, K_s$ of the conjugacy classes of maximal abelian subgroups of $G$. For nonidentity $x \in K_i$, we note that $C_G(x)$ is abelian, and that since $K_i$ is abelian, $K_i \subset C_G(x)$. But $K_i$ is maximal, so $C_G(x) = K_i$. We see that $K_i$ satisfies the properties of Lemma 2.5, and so $K_i$ is a *TI*-subset of $G$. Set $H_i = N_G(K_i)$. If $K_i = H_i$, then $(H_i \cap H_i^x) = \{1\}$ for $x \in (G - H_i)$ by Lemma 2.1, and so $H_i$ is a Frobenius subgroup and $G$ a Frobenius group. So we assume $G$ is nonabelian and that $K_i \neq H_i\ \forall i$ and derive a contradiction.

Let $l_i = |H_i : K_i|$ and $w_i = (|K_i| - 1)/l_i$. We know by Lemma 2.5 that there exists a set of $w_i$ irreducible characters on $H_i$ of like degree which vanish outside of $K_i$. Since $|G|$ is odd, so too are $l_i$ and $|K_i|$, and so $w_i$ must be even and hence $\geq 2$. So we employ Theorem 2.3 to construct a family $E_i$ of $w_i$ exceptional characters on $G$.

Note that for $i \neq j$, nonidentity elements of $K_i$ and $K_j$ are not conjugate, since $x \in (K_i - \{1\})$ and $x^y \in (K_j - \{1\}) \implies K_i^y = (C_G(x))^y = C_G(x^y) = K_j$, contradicting the fact that $K_i$ and $K_j$ are not conjugate. Hence, each $K_i$ contains $w_i$ nontrivial conjugacy classes distinct from those of any $K_j$, $j \neq i$. Since any element of $G$ must be contained in some maximal abelian subgroup, we see that $G$ has precisely $\sum w_i + 1$ conjugacy classes as a result, and hence there are $\sum w_i + 1$ irreducible representations on $G$.

But for each $K_i$, we have defined a family $E_i$ of $w_i$ exceptional characters on $G$. If $\chi_1$ and $\chi_2$ are distinct exceptional characters for $K_i$, we can easily verify from the definition of the induced character that $\chi_1 = \chi_2$ on $K_j$, $j \neq i$, since the irreducible characters on $H$ from which $\chi_1$ and $\chi_2$ were induced must vanish outside of $K_i$. So if $\chi_1$ were also an exceptional character for some $K_j$, $\chi_2$ would be an exceptional character for $K_j$ likewise, since $\chi_1$ and $\chi_2$ are identically defined on $K_j$. But then $\chi_1 = \chi_2$ on $K_i$, contradicting the distinctness of $\chi_1$ and $\chi_2$. Thus, the $\sum w_i$ exceptional characters so far defined are all unique. We see that the $\sum w_i$ nontrivial irreducible characters on $G$ must all be exceptional characters for some $K_i$. But then, by Lemma 2.4, $K_i = N_G(K_i) = H_i$ for some $i$. Contradiction. □

## References

1. Michio Suzuki, *Group theory I*, Springer-Verlag, New York, US, 1982.
2. ______, *Group theory II*, Springer-Verlag, New York, US, 1986.