Math 126, sheet 3

February 25, 2000

All representations here are over the field $\mathbf{C}$.

Problem 1. Let G be a finite group having a representation on a vector space V. For each $k \geq 0$, we can form the k-fold tensor product, $V^{\otimes k} = V \otimes V \otimes \cdots \otimes V$ (k factors). By convention, the tensor product with zero factors is the trivial 1-dimensional representation.

Now suppose V is faithful and let W be any irreducible representation of G. Show that W occurs with non-zero multiplicity in the tensor product $V^{\otimes k}$ for some $k \geq 0$.

Problem 2. Let G be a group and let $V_1, \ldots, V_r$ be its irreducible representations. Let W be any representation. Consider the decomposition of $W \otimes V_i$ into irreducibles:

$$W \otimes V_i = \bigoplus_{j=1}^{r} m_{ij} V_j,$$

(i.e. V_j occurs with multiplicity m_{ij} in the tensor product.) If $\bar{W} \cong W$, show that the $r \times r$ matrix $M = (m_{ij})$ is symmetric.

Show that the columns of the character table of G are a basis of eigenvectors for the matrix M.

Let G be S_5, and let $W = W_4$ be the 4-dimensional representation obtained as usual by decomposing the 5-dimensional permutation representation as $V = U \oplus W_4$. Let M be the matrix corresponding to this G and W, as defined above. What is the rank of M?

Problem 3. Let G be a finite group having a *faithful* 2-dimensional representation W, and corresponding matrix representation A. Suppose that $\det A(g) = 1$ for all g.

Let M be the matrix formed from G and W as above. Show that M is a symmetric matrix with non-negative integer entries, all of whose eigenvalues lie in the range $[-2, 2]$. Show that 2 is an eigenvalue of M and that its multiplicity is 1 (i.e. it does not occur as a repeated root of the characteristic polynomial). Show that if -2 is an eigenvalue of M then the order of G must be even. Show that -2 cannot be a repeated eigenvalue.

If the matrix M turns out to be

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then what are the degrees of the irreducible representations of G? What is the order of the group?

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