Math 126, sheet 7

April 15, 2000

Question 1. Let V be the space of continuous functions $f : \mathbf{R} \to \mathbf{C}$. If $f \in V$ and $t \in \mathbf{R}$, let $\tau_t f$ denote the function obtained by translating f by t:

$$(\tau_t f)(x) = f(x - t).$$

In this way, V becomes a representation space for the group $G = (\mathbf{R}, +)$.

Suppose $f \in V$ is smooth and satisfies a constant-coefficient, linear differential equation:

$$f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' + a_0 f = 0.$$

Show that f belongs to finite-dimensional G-invariant subspace of V.

Now suppose H is a finite-dimensional linear subspace of V that is invariant under G. Suppose that all functions in H are smooth. Show that there is a differential equation of the above sort satisfied by all elements of H.

Optional. Drop the hypothesis of smoothness from the last part: show that any finite-dimensional invariant subspace H of V necessarily consists only of smooth functions.

Question 2. Let V be the space of continuous functions on the unit sphere S^2, and let G be the group $SO(3)$ acting on V. Show that a function $f \in V$ is contained in a finite-dimensional G-invariant subspace if and only if f can be expressed as a polynomial in the coordinates x, y, z.

Question 3. Let G be a group with generators S, T and U subject only to the relations

$$SU = US, \quad TU = UT, \quad STS^{-1}T^{-1} = U.$$

(The first two tell us that U is in the center of G. The group G is infinite!)

Let $\mathcal{V}$ be the vector space of all functions $\mathbf{R} \to \mathbf{C}$. Fix a non-zero $\mu \in \mathbf{C}$. Show how to make $\mathcal{V}$ into a G-module in such a way that the generator U acts by $Uf = \mu f$ for all $f \in \mathcal{V}$. If μ happens to be a root of unity, show that $\mathcal{V}$ contains a non-zero, finite-dimensional submodule. (Note that $\mathcal{V}$ consists of all functions, not just the continuous ones.)

Now suppose μ is not a root of unity. Show that there is no finite-dimensional complex representation V of the group G having the property that $Uv = \mu v$ for all $v \in V$.

Show that if μ is the root of unity $e^{2\pi i/n}$ and V is a finite-dimensional complex representation of G with the property that $Uv = \mu v$ for all $v \in V$, then the dimension of V is at least n.

Hint for the last two parts: every complex linear transformation on a finite-dimensional vector space has an eigenvector.

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