Solution Set 8

1. O(2) is generated by a single reflection R and rotations ρ_θ. Note the relation $R\rho_\theta = \rho_{-\theta}R$. Every element of O(2) can be described as ρ_θ or $R\rho_\theta$ for some θ. The following is a list of all the irreducibles:

trivial

A(rotation)$= 1$, A(reflection)$= -1$

For each positive integer k, $B_k(\rho_\theta) = \begin{pmatrix} e^{ik\theta} & 0 \\ 0 & e^{-ik\theta} \end{pmatrix}$, $B_k(R) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

It's easy to see that these representations are irreducible, and that these representations are inequivalent. We need to show that this list is a complete list. Let C be any irreducible representation of O(2) with representation space V. Consider the representation of SO(2) that we get by restriction. Since we know what the irreducibles of SO(2) are, we know that there is a k and a $v \in V$ such that $C(\rho_\theta)v = e^{ik\theta}v$ for all θ. Observe that the space spanned by v and $C(R)v$ is invariant, so V must be spanned by these two vectors.

If dim $V = 1$, then $C(R) = \pm 1$, and $C(\rho_\theta) = C(\rho_{-\theta})$. That is, $e^{ik\theta} = e^{-ik\theta}$ for all θ. Thus $k = 0$, so C is either trivial or it's A.

Now suppsose dim $V = 2$. Since the determinant of this representation gives us a one-dimensional representation of O(2), it follows that $\det(C(\rho_\theta)) = 1$. So $C(\rho_\theta)$ has eigenvalues $e^{ik\theta}$ and $e^{-ik\theta}$. It follows from our discussion above that writing C in the basis $v, C(R)v$ yields B_k for some k.

2.(a) Let η_i be the character of $A^{(i)}$ and η the character of A. Then $\eta = a_1\chi_1 + \cdots + a_k\chi_k$ where the χ_j's are irreducible characters. Since for each $g \in G$, $A^{(i)}(g) \longrightarrow A(g)$, we have that $\langle \eta_i, \chi_j \rangle \longrightarrow \langle \eta, \chi_j \rangle$ for each j. Note that this follows from the compactness of G. For a given j, $\langle \eta_i, \chi_j \rangle$ is a converging sequence of integers, and therefore it must stabilize. Finally, for sufficiently large i, η_i cannot contain any extraneous irreducible characters (that is, ones not occurring in η) because we can count the degrees.

(b) Consider the representations of the additive group of real numbers given by $A^{(m)}(x) = e^{x/m}$.

3. There exists $k < n$ such that we can pass to a subsequence $A^{(i)}$ where each $A^{(i)}$ has an invariant subspace W_i of dimension k. Now let C^k be a fixed k-dimensional subspace of the larger space C^n. There exists $U_i \in$SU(n) such that U_i maps C^k bijectively to W_i. It follows that C^k is an invariant subspace for the representation $g \mapsto U_i^{-1}A^{(i)}(g)U_i$. Since SU$(n)$ is compact, U_i has a convergent subsequence. Without changing notation, assume we've passed to this subsequence. So $U_i \to U$ for some $U \in$SU(n). We claim that C^k is an invariant subspace for the representation $g \mapsto U^{-1}A(g)U$. To see this, observe that for each g, $U^{-1}A(g)U = \lim U_i^{-1}A^{(i)}(g)U_i$ which is a limit of matrices that

look like

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

Hence, A is reducible.

To see that the converse is false, let G be the group of 2 by 2 matrices under addition. Then $A^{(m)}(M) = e^{M/m}$ provides a counterexample.

The second part of the problem is false—the complete reducibility of the $A^{(i)}$'s does not imply the complete reducibility of A. Here's a counterexample:

Let G the real numbers under addition. Define

$$A^{(m)}(x) = \begin{pmatrix} e^{x/m} & m(e^{x/m} - 1) \\ 0 & 1 \end{pmatrix}$$

which is equivalent to the direct sum of the trivial representation and the representation $x \mapsto e^{x/m}$. Taking the limit, we find that

$$A(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

which is not completely reducible.